Ginzburg-Landau minimizers in perforated domains with prescribed degrees

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Abstract.

Suppose that Ω is a 2D domain with holes $\omega_0, \omega_1, \ldots, \omega_j, j = 1...k$. In the perforated domain $A = \Omega \setminus (\cup_{j=0}^{k} \omega_j)$ we consider the class $\mathcal{J}$ of complex valued maps having degrees 1 and -1 on the boundaries $\partial\Omega$, $\partial\omega_0$ respectively and degree 0 on the boundaries of other holes.

We investigate whether the minimum of the Ginzburg-Landau energy E_κ is attained in $\mathcal{J}$, as well as the asymptotic behavior of minimizers as the coherency length κ^{-1} tends to 0. We show that the answer to these questions is determined by the value of the H^1-capacity cap(A) of the domain. If cap$(A) \geq \pi$ (domain A is "thin"), minimizers exist for each κ. Moreover they are vortexless and converge in $H^1(A)$ (and even better) to a minimizing S^1-valued harmonic map as $\kappa \to \infty$. When cap$(A) < \pi$ (domain A is "thick"), we establish existence of quasi-minimizers (maps with "almost minimal energy"), which exhibit a different qualitative behavior : they have exactly two zeroes (vortices) rapidly converging to ∂A as $\kappa \to \infty$. Finally we formulate a conjecture on non-existence of the minimizers in thick domains.

Update. This preprint was written at the end 2004. We added (in June 2008) a short final section accounting subsequent developments.

1 Introduction

Our study is motivated by the following problem. In [4], [5], a mathematical model of an **ideal** superconductor reinforced by a large number of thin insulating rods was introduced. For a cylindrical superconductor with coaxial cylindrical hole (often used in experimental settings), this model led to a minimization problem for **harmonic maps** in a 2D annular domain with many small holes. The distinguishing mathematical feature of this problem is that the physical insulating conditions lead to prescribing degree (winding number) boundary conditions. Even though this problem is nonlinear, it has an underlying **linear** problem for the multi-valued phase of the harmonic maps, which is why existence of the minimizers for any fixed number of holes is trivial and the main issue addressed in [4], [5] was the homogenization limit when number of holes tends to infinity.

This study led to a natural question : what if the superconductor in the composite described below is not ideal (e.g., of type II) ?

Mathematically, this means that, in the above minimization problem, the Dirichlet integral for harmonic maps should be replaced by the Ginzburg-Landau (GL) functional. Then the existence question becomes highly nontrivial and it leads to the following problem

$$m_\kappa = \mathrm{Inf}\Big\{E_\kappa(u) = \frac{1}{2}\int_A |\nabla u|^2 + \frac{\kappa^2}{4}\int_A (1-|u|^2)^2 \ ; \ u \in \mathcal{J}\Big\}. \tag{1.1}$$

Here, E_κ is a GL type energy (without magnetic field), A is a 2D perforated domain, i.e.

$$A = \Omega \setminus (\cup_{j=0}^{k} \omega_j), \quad \overline{\omega_j} \subset \Omega, \ j = 0, \dots, k, \ \overline{\omega_j} \cap \overline{\omega_l} = \emptyset, \ j \neq l, \tag{1.2}$$

with Ω, ω_j, $j = 0, \dots, k$, simply connected bounded smooth domains. The class $\mathcal{J}$ of testing maps is

$$\begin{aligned} \mathcal{J} = \{u \in H^1(A\ ; I\!\!R^2);\ |u| = 1 \text{ a.e. on } \partial A, \deg(u, \partial\Omega) = 1,& \\ \deg(u, \partial\omega_0) = -1, \deg(u, \partial\omega_j) = 0,\ j = 1, \dots, k\}.& \end{aligned} \tag{1.3}$$

We thus consider a domain with finitely many fixed holes ω_j. The constant κ^{-1} is the coherency length (GL parameter). For the sake of our discussion, we will allow κ to be 0, so that throughout this paper we let $\kappa \geq 0$.

A point that needs clarification is whether the definition of $\mathcal{J}$ is meaningful. In other words, we discuss whether, given a map $u \in H^1(A)$ such that $|u| = 1$ a.e. on ∂A, we can define the degree of u on each component of ∂A. For this purpose, we start by briefly recalling the definition and the basic properties of the degree of a continuous complex-valued map (see, e.g., [1]). Let Γ be a C^1 simple closed curve in $\mathbb{C}$. Intuitively speaking, the degree of a map $v \neq 0$ is defined as follows. Suppose that the image of Γ, $v(\Gamma)$, is a closed curve that surrounds the origin. If we cover Γ once, then $v(\Gamma)$ winds around the origin a number of times, either in the positive direction (counterclockwise), or in the negative direction (clockwise). Then the degree of v is the number of positive loops minus the number of negative loops around the origin. This integer depends on the sense we choose to cover Γ. To give the formal definition we assume Γ to be oriented, i.e. we consider a parametrization $f : [0, 1] \to \Gamma$, with $f(0) = f(1)$. Let $v : \Gamma \to \mathbb{C}$ be a continuous map and set $w = v \circ f$. It is a simple exercise that, if v is always different from 0 on Γ, then we may write, on $[0, 1]$, $w(t) = |w(t)|e^{\imath\varphi(t)}$ for some continuous φ. Since $w(0) = w(1)$, it follows that $\varphi(1) - \varphi(0) \in 2\pi Z\!\!\!Z$. The integer $d = \dfrac{\varphi(1) - \varphi(0)}{2\pi}$ is called **the degree of v with respect to 0** ; it is denoted by $\deg(v, \Gamma, 0)$, or $\deg(v, \Gamma)$, or $\deg v$. The definition is meaningful, in the sense that the value of d does not change if we replace f by another parametrization g which yields the same orientation on Γ.

More generally, if $a \in \mathbb{C}$ is not among the values of v, then we may define $\deg(v, \Gamma, a)$ as $\deg(v-a, \Gamma, 0)$. It is easy to see that, if v has more regularity, say $v \in C^1$, then

$$d = \frac{1}{2\imath\pi}\int_\Gamma \frac{1}{v}\frac{\partial v}{\partial \tau} = \frac{1}{2\imath\pi}\int_\Gamma \frac{\overline{v}}{|v|^2}\frac{\partial v}{\partial \tau}, \tag{1.4}$$

where τ is the tangent vector directly oriented with respect to the fixed orientation. In particular, if we change orientation on Γ, d changes to $-d$. A special case of the above formula is obtained when $|v| = 1$, i.e., when $v \in C^1(\Gamma; S^1)$. In this case, we have

$$\deg\ v = \frac{1}{2\pi}\int_\Gamma v \wedge \frac{\partial v}{\partial \tau}, \tag{1.5}$$

where we have used the following

Notation. If $z = a + \imath b, w = c + \imath d \in \mathbb{C}$, then $z \wedge w = ad - bc$. We will also use later the following notation : if u, v are complex-valued maps, then $u \wedge \nabla v = \begin{pmatrix} u \wedge \dfrac{\partial v}{\partial x} \\ u \wedge \dfrac{\partial v}{\partial y} \end{pmatrix}$.

We quote here the main properties of the degree of non vanishing maps :
a) if

$$|v - w| < \mathrm{Min}\{|v(z)|,\ |w(z)|\ ;\ z \in \Gamma\},$$

then deg v= deg w. In particular, the degree is continuous with respect to uniform convergence ; b) $\deg(vw)$ =deg v+deg w ; c) deg $\overline{v} = -$deg v ; d) if $a, b \in \mathbb{C}$ lie in the same connected component of $\mathbb{C} \setminus v(\Gamma)$, then deg (v, Γ, a) =deg (v, Γ, b) ; e) if $v \in C^1(S^1; S^1)$ has the Fourier expansion $v = \sum_{n\in\mathbb{Z}} a_n e^{\imath n\theta}$, then

$$\deg\ v = \sum_{n\in\mathbb{Z}} n|a_n|^2. \tag{1.6}$$

Assume now that $v \in H^{1/2}(\Gamma)$. Then the right-hand side of (1.6) makes sense ; so does the right-hand side of (1.5) if we interpret it appropriately, i.e., if we write it as

$$\frac{1}{2\pi}\Big(< v_1, \frac{\partial v_2}{\partial \tau} >_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)} - < v_2, \frac{\partial v_1}{\partial \tau} >_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)} \Big).$$

This suggests the following

Definition 1 ([12]) *Let $v \in H^{1/2}(\Gamma; S^1)$. Then*

$$\deg\, v = \frac{1}{2\pi}\Big(< v_1, \frac{\partial v_2}{\partial \tau} >_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)} - < v_2, \frac{\partial v_1}{\partial \tau} >_{H^{1/2}(\Gamma), H^{-1/2}(\Gamma)} \Big). \tag{1.7}$$

Warning. In order to keep the notations simple, we will write, in what follows, the degree formula in the form (1.5), even when v is only in $H^{1/2}$.

The surprising feature of this degree of $H^{1/2}$ maps is that it is still an integer ; this was proved by L. Boutet de Monvel, see [12]. The degree is continuous with respect to $H^{1/2}$ convergence (see [20]), which is the extension of property a) to $H^{1/2}$ maps. Properties b) and c) are still valid for $H^{1/2}$ maps ; see Section 2 and Appendix A. There is an analogue of property d), but it is more delicate to state (and not used in this paper) ; we send the reader to [20] for details. On the other hand, (1.6) clearly holds when $v \in H^{1/2}$, by the definition of the degreee. Finally, we note that the definition of degree still depends on the orientation we choose on Γ !!since, if we change orientation, then $\dfrac{\partial v}{\partial \tau}$ changes to $-\dfrac{\partial v}{\partial \tau}$.

One can, more generally, define the degree of a map $v \in H^{1/2}(\Gamma; \mathbb{C})$ provided its range is far away from 0. More specifically, assume that there exist constants $a, b > 0$ such that $a \le |v| \le b$ a.e. on Γ. Then we set

$$\deg\, v = \deg\, \frac{v}{|v|}. \tag{1.8}$$

We now return to the definition of $\mathcal{J}$. Let $u \in H^1(A)$ be such that $|u| = 1$ a.e. on ∂A and set $v = \mathrm{tr}_{|\partial A} u$. For each connected component Γ of ∂A, we have $v \in H^{1/2}(\Gamma; S^1)$ and thus we may define the degree of v on Γ, provided we choose an orientation on Γ. Throughout this paper, we use the following convention : each component Γ of ∂A is oriented with the direct orientation with respect to A. The degrees we prescribe in the definition of $\mathcal{J}$ are the degrees of v computed with respect to this orientation. Thus, for example, if $A = \{z\,;\, \rho < |z| < R\}$ and $u(z) = z/|z|$, then $\deg(u, \partial\omega_0) = -1$ and $\deg(u, \partial\Omega) = 1$. On the other hand, recall that each simply closed planar curve Γ has a natural orientation (counterclockwise). With our convention, given a general domain A, the orientation of $\partial\Omega$ is the natural one, while the orientation of $\partial\omega_j$, $j = 0, \dots, k$, is the opposite of the natural one.

We complete our discussion of degree by mentioning another basic property of the degree of continuous maps : f) Assume that $u \in C(\overline{A}; \mathbb{C})$ is such that $u \neq 0$ on ∂A. Assume also that $\deg(u, \partial\Omega) + \sum_{j=0}^{k} \deg(u, \partial\omega_j) \neq 0$. (Here,the orientation on ∂A is direct with respect to A.) Then u has has (at least) a zero in A. There is an analogue of f) for H^1-maps, but the statement is more subtle ; see [20]. We will prove in Appendix B a weak analogue of f) sufficient for our needs.

We may now address a first natural question concerning the minimization problem (1.1)-(1.3)

Question 1. Is m_κ attained ?

Before discussing this question, we start by recalling the most intensively studied minimization problem for the Ginzburg-Landau functional (see [10]), namely

$$e_\kappa = \mathrm{Inf}\{E_\kappa(u)\,;\,u \in \mathcal{L}\}, \tag{1.9}$$

where

$$\mathcal{L} = \{u \in H^1(G)\,;\,\mathrm{tr}_{\partial G} u = g\}. \tag{1.10}$$

Here, G is a smooth bounded domain in $I\!R^2$ and $g \in H^{1/2}(\partial G; S^1)$ is fixed. In this case, the minimum is obviously attained in (1.9). The reason is that $\mathcal{L}$ is closed with respect to weak H^1 convergence ; therefore, if we take a minimizing sequence for (1.9)-(1.10) that weakly converges to some u, this u is in $\mathcal{L}$ and clearly minimizes (1.9)-(1.10).

The situation is more delicate when we do not prescribe a Dirichlet boundary condition, but only degrees, as shown by the following

Example 1 Let

$$n_\kappa = \mathrm{Inf}\{E_\kappa(u)\,;\,u \in \mathcal{M}\}, \tag{1.11}$$

where

$$\mathcal{M} = \{u \in H^1(I\!D)\,;\,|u| = 1 \text{ a.e. on } S^1\,,\ \deg(u, S^1) = 1\}. \tag{1.12}$$

Here, $I\!D$ is the unit disc and we consider the natural orientation on S^1. Then, for each $\kappa > 0$, $n_\kappa = \pi$ and n_κ is not attained.

We will prove and extend this example in Section 4. In particular, this example implies that the class $\mathcal{M}$ is not closed with respect to weak H^1 convergence (it is closed with respect to strong H^1 convergence since degree is continuous for the strong $H^{1/2}$ convergence). Here is an example of sequence in $\mathcal{M}$ weakly converging in H^1 to a map which is not in $\mathcal{M}$:

Example 2 Let $(a_n) \subset (0,1)$ be such that $a_n \to 1$. Set $u_n(z) = \dfrac{z - a_n}{1 - a_n z}$, $z \in I\!D$. Then $u_n \rightharpoonup -1$ weakly in H^1.

Clearly $u_n \to -1$ a.e. (weak H^1 convergence will be established in Section 4). Example 2 is adapted in Section 4 to the class $\mathcal{J}$ in order to prove the following

Proposition 1 *The class $\mathcal{J}$ is not closed with respect to weak H^1-convergence.*

This implies that the existence of minimizers of (1.1)-(1.3) does not follow immediately from the direct method of the Calculus of Variations.

Before discussing Question 1 further, we mention some useful a priori bounds on m_κ. Recall that in the case of a prescribed Dirichlet data with non zero degree (thoroughly studied in [10]) the Ginzburg-Landau energy tends to infinity as $\kappa \to \infty$. However, a straightforward calculation shows that the energy remains bounded when we have only specified degrees on the boundary. More specifically, in Section 6 we prove that, for degrees 1, -1, $0, \ldots, 0$, we have

$$m_\kappa \leq 2\pi. \tag{1.13}$$

Note that the right-hand side of (1.13) is independent of A and κ. In fact, our construction yields analogous upper bounds for arbitrary degrees.

Let us briefly sketch how this upper bound is obtained. We want to consider $u \equiv 1$ **as** simplest testing map, however, this u has to be modified in order to satisfy the required degree conditions. The modified u can be described as follows :
(i) u equals 1 in the domain $A \setminus (D \cup \Delta)$ (see Fig. 1 below) ;
(ii) on the boundaries of D and Δ, u has modulus 1 and degrees -1, 1 respectively (see Fig. 2) ;
(iii) u is harmonic in D and in Δ.
By choosing appropriately the phases φ and ψ in Picture 2, we prove that $E_\kappa(u) \to 2\pi$ as D and Δ shrink to points.
Roughly speaking, such a testing function has a "vortex" (zero of degree -1 or 1) in D and in Δ, and the energy of each vortex is almost π. We will call these functions "vortex testing maps".

There is yet another upper bound, which is obtained by considering all the possible testing maps of modulus 1 in A. More specifically, we consider the class

$$\mathcal{K} = \{u \in \mathcal{J}\,;\, |u| = 1 \text{ a.e. in } A\}. \tag{1.14}$$

It turns out that $\mathcal{K}$ is not empty because the degrees we prescribe have total sum 0 ; see Lemma 2.2 below. It is known that, in $\mathcal{K}$, the minimum of the Ginzburg-Landau energy is attained (see [10]). Let

$$I_0 = \text{Min } \{E_\kappa(u)\,;\, u \in \mathcal{K}\} = \text{Min } \left\{\frac{1}{2}\int_A |\nabla u|^2\,;\, u \in \mathcal{K}\right\}. \tag{1.15}$$

Then, clearly,

$$m_\kappa \leq I_0. \tag{1.16}$$

A more delicate property (see Section 6) is

$$m_\kappa < I_0\,. \tag{1.17}$$

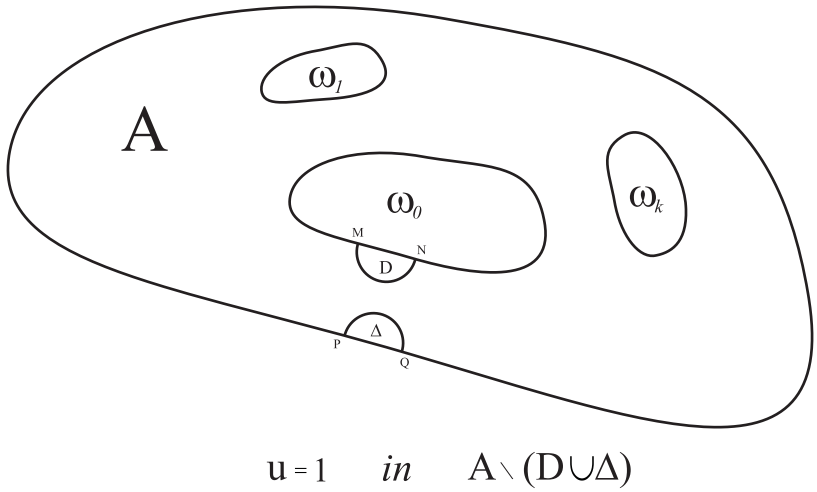

Figure 1: u equals 1 except on D and Δ

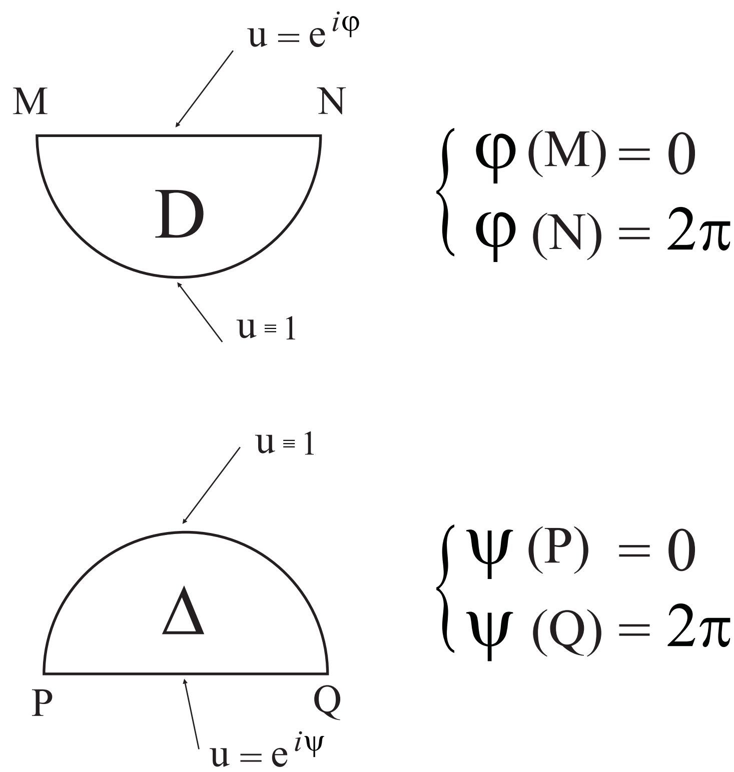

Figure 2: On $\partial D \cap \partial A$, the phase of u jumps 2π form M to N. On $\partial\Delta \cap \partial A$, the jump is from P to Q

Clearly, (1.13) and (1.17) imply that $m_\kappa \le \mathrm{Min}\,\{I_0, 2\pi\}$. This bound is close to optimal when κ is large. Indeed, we prove in Sections 6 and 7 that, for any A, we have

$$\lim_{\kappa\to\infty} m_\kappa = \mathrm{Min}\,\{I_0, 2\pi\}. \tag{1.18}$$

It turns out that I_0 can be expressed in explicit geometrical terms using Newtonian capacity of the domain A ; this is done in Sections 2 and 3. Here is a very simple example that will be detailed and generalized in Section 2 :

Example 3 Let $A = \{z\,;\,\rho < |z| < R\}$. Then the H^1-capacity of A is $\mathrm{cap}(A) = \dfrac{2\pi}{\ln(R/\rho)}$ and

$$I_0 = \frac{2\pi^2}{\mathrm{cap}(A)}. \tag{1.19}$$

In Section 2, we show that (1.19) is valid for any domain of the form $\Omega \setminus \omega_0$. Moreover, we introduce a generalized capacity such that (1.19) still holds for an arbitrary perforated domain $A = \Omega \setminus (\cup_{j=0}^k \omega_j)$. In Section 3, we provide yet another geometrical interpretation of I_0 in terms of conformal representations.

Formula (1.18) suggests that one has to distinguish between three types of domains :
a) "subcritical", for which $I_0 < 2\pi$ (or, equivalently, $\mathrm{cap}(A) > \pi$) ;

b) "critical", for which $I_0 = 2\pi$ (or $\mathrm{cap}(A) = \pi$) ;

c) "supercritical", for which $I_0 > 2\pi$ (or $\mathrm{cap}(A) < \pi$).

This terminology is motivated by our results concerning the existence and behavior of minimizers, that we discuss below. We illustrate a)-c) by using Example 3. When A is a circular annulus, a) corresponds to $R/\rho < e^2$, b) to $R/\rho = e^2$ and c) to $R/\rho > e^2$. Intuitively, one should think of subcritical domains as "thin" domains, and of supercritical domains as "thick" domains. This is obvious for a circular annulus. For a generic domain, this follows from the geometrical interpretation of H^1-capacity.

We now return to the existence of minimizers. The main tool in proving existence is the following result, established in Section 5

Proposition 2 *Assume that $m_\kappa < 2\pi$. Then m_κ is attained.*

The first result of this type was established for the Yamabe problem by Th. Aubin in [3]. Such results subsequently proved to be extremely useful in minimization problems with possible lack of compactness of minimizing sequences ; see [19], [16], [17], [13] and the more recent papers [18] and [23].

The proof of Proposition 2 relies on the following "Price" Lemma

Lemma 1 *Let (u_n) be a bounded sequence in $\mathcal{J}$. Assume that $u_n \rightharpoonup u$ weakly in $H^1(A)$ (and thus we must have $|u| = 1$ a.e. on ∂A). Then :*

$$\liminf \frac{1}{2}\int_A |\nabla u_n|^2 \geq \frac{1}{2}\int_A |\nabla u|^2 + \pi\Big(|1-\deg(u,\partial\Omega)| + |-1-\deg(u,\partial\omega_0)| + \sum_{j=1}^{k} |\deg(u,\partial\omega_j)|\Big) \quad (1.20)$$

and

$$\frac{1}{2}\int_A |\nabla u|^2 \geq \pi\Big|\deg(u,\partial\Omega) + \sum_{j=0}^{k} \deg(u,\partial\omega_j)\Big|. \quad (1.21)$$

The proof of Lemma 1 is presented in Section 5. The argument there works for arbitrary fixed degrees instead of $1, -1, 0, \ldots, 0$. Intuitively, the estimate (1.20) shows that the minimal energy needed to jump from degree d (for the maps u_n) to degree δ (for u), on a component of ∂A, is $\pi|d-\delta|$. This "price" is, in general, optimal as shown by the maps in Example 2.

As an immediate consequence of Proposition 2 and of the upper bound (1.17), we obtain the following

Theorem 1 *Assume that A is subcritical or critical. Then m_κ is attained for each $\kappa \geq 0$.*

Remark 1 Minimizers of (1.1)-(1.3), whenever they exist, are smooth. This requires some proof, since minimizers or, more generally, critical points of E_κ in $\mathcal{J}$, satisfy mixed type boundary conditions : Dirichlet for the modulus, and Neumann for the phase. Smoothness of critical points is established in Appendix C. The discussion on the degree of $H^{1/2}$ maps is not essential for the understanding of our proofs. The main ideas can be understood by considering smooth maps in (1.1)-(1.3).

In the subcritical and critical cases, we further address the following natural

Question 2. What is the behavior of minimizers u_κ of (1.1)-(1.3) as $\kappa \to \infty$?

The answer is given by

Theorem 2 *Assume that A is subcritical or critical. Let u_κ be a minimizer of (1.1)-(1.3). Then, up to a subsequence, $u_\kappa \to u_\infty$ in $C^{1,\alpha}(\overline{A})$, $\forall\ 0 < \alpha < 1$. Here, u_∞ is a minimizer of (1.14)-(1.15).*

Remark 2 It is known that, for the GL equation, one can not improve the $C^{1,\alpha}$ convergence to, say C^2 convergence (H. Brezis, personal communication).

When A is subcritical, the proof of Theorem 2 relies on a straightforward adaptation of the arguments developed in [9], combined with (1.17). The critical case is much more subtle. Note that Theorem 2 implies that, for large κ, $|u_\kappa| \approx 1$, that is the minimizers are "vortexless".

We now turn to the supercritical case, i.e., we assume $I_0 > 2\pi$. In this case, our analysis is less complete, in particular, we were not able to determine whether the value m_κ is attained or not in $\mathcal{J}$. However, a simple consequence of Proposition 2 is that, when A is fixed and κ varies from 0 to ∞, there are only three possibilities

Theorem 3 *Assume that A is supercritical. Then (exactly) one of the three following possibilities holds :*
a) m_κ is attained for all $\kappa > 0$;
b) m_κ is never attained ;
c) there is some $\kappa_1 \in (0, \infty)$ such that : if $\kappa < \kappa_1$, then m_κ is attained, while if $\kappa > \kappa_1$, then m_κ is not attained.

Which one of the possibilities a), b) or c) actually occurs for a given A remains at present an open question. However, when A is of the form $\Omega \setminus \omega_0$, we were able to rule out possibility b) :

Proposition 3 *Assume that $A = \Omega \setminus \omega_0$. Then either a) or c) holds.*

By a formal analysis, we believe that possibility a) **never** occurs, and we were thus led to the following

Conjecture. *Assume that A is supercritical. Then there is a constant $\kappa_1 \geq 0$ such that, if $\kappa > \kappa_1$, then m_κ is never attained.*

Since we do not know whether, for large values of κ, there are minimizers of (1.1)-(1.3), we are led to consider "quasi-minimizers". These maps, which are defined at the beginning of Section 7, are solutions of the Ginzburg-Landau equation with almost minimal energy. The advantage of considering quasi-minimizers is that they do always exist, and that any minimizer of (1.1)-(1.3) (if it exists) is a quasi-minimizer. The relevant difference between the subcritcal/critical case and the supercritical case is that, in the supercritical case, quasi-minimizers develop "vortices" for large values of κ.

The notion of a vortex is not clearly defined in the GL literature (although it is perfectly understood !). Here we discuss briefly the notion of a vortex for solutions of the 2D GL equation

$$-\Delta u_\kappa = \kappa^2 u_\kappa (1 - |u_\kappa|^2) \quad \text{in } A. \tag{1.22}$$

The role of this discussion is to clarify different possible meanings of a vortex, although we will not give its formal definition.

(i) The most common understanding is that a "vortex of u_κ" is a zero z of u_κ. A bit more restrictive definition requires in addition : a) that z is an isolated zero ; b) that the degree of u_κ computed on small circles around z is different from 0. Condition a) is not too restrictive, however. Indeed, most of the time one considers minimizers of the Ginzburg-Landau energy with respect to some

Dirichlet boundary data that does not vanish on the boundary ; in this case, all zeroes of u_κ are isolated, see [24]. It is common belief that condition b) is not restrictive either, in the sense that, for **large** κ, zeroes of degree 0 cease to exist. This is not known in general, but it is proved in many situations ; in particular, this holds for minimizers of the Ginzburg-Landau energy with respect to a fixed boundary data of modulus 1, see [10].

Here, the vortices are defined intrinsically by the u_κ's. Note that, in this setting, the vortices are **not** singularities, since each u_κ is smooth.

(ii) There is however another perspective, when a vortex can be defined as a singularity of a map. Suppose that a given sequence $u_\kappa \in H^1$ converges (up to subsequences) to some map u at least a.e. Then one could define a "vortex of u" as a singularity of u (u is not smooth near this point). Note that, in this setting, one has to consider asymtotic behavior of the sequence u_κ's as $\kappa \to \infty$, since it determines u.
It is a common belief that the vortices of u are related to the vortices of the u_κ (defined above), as follows : given a vortex z of u, there are, for large κ, vortices z_κ of u_κ (i.e., zeroes of u_κ) such that $z_\kappa \to z$. This property is not proved in all the possible situations, but it is known to hold in many cases ; in particular, for a fixed boundary data ([10]). The converse is known to be false, i.e., vortices z_κ of u_κ need not approach a vortex of u ; for example, if the z_κ's "escape to the boundary". Note, that while (i) describes vortices of smooth functions (solutions of GL PDE are smooth), (ii) introduces vortices of functions, which are not necessarily smooth.

(iii)

A different perspective is to start by considering "regular points" of (u_κ). Suppose that (u_κ) is a family of functions in H^1 such that $u_\kappa \to u$ strongly in H^1 in some neighborhood of a point z. Then z is called a regular point of the the family (u_κ).
One expects that a point in A is a vortex of u (in the sense of (ii))) if and only if it is not a regular point of (u_κ) ; this need not be true for points z on ∂A. This result was rigorously proved in [10] for all the points in $\overline{A}$, when the boundary data is fixed ; in other words, in that context, a point in $\overline{A}$ is regular if and only if it is not an accumulation point of vortices z_k of the u_κ's. This property is also known to hold, for points $z \in A$, in many other situations.

(iv) There is a fourth point of view, which is particularly useful when treating the Ginzburg-Landau equation in presence of the magnetic field or the 3D Ginzburg-Landau equation ; this point of view was first developed in [35] and [31]. Loosely speaking, a point $z \in A$ is a "concentration point (for (u_κ))" if there is some $C > 0$ such that, for any neighborhood U of z, the energy of u_κ in U is at least C for large values of κ. The energy considered in this approach is usually the GL energy, possibly rescaled by an appropriate factor.
Concerning concentration points $z \in \overline{A}$, the are two rigorous results one expects :
a) z is a concentration point (for (u_κ)) if and only if z it is not a regular point (for (u_κ) ;
b) z is a concentration point (for (u_κ)) if and only if there are vortices z_κ of u_κ that tend to z.

Note that, for a point $z \in A$, one expects

$$z \text{ vortex} \iff z \text{ limit of vortices of } u_\kappa \iff z \text{ not a regular point} \iff z \text{ concentration point.}$$

We may now state in an informal way the results we establish in Section 11 relative to the properties of the quasi-minimizers u_κ for **large** values of κ in the supercritical case $I_0 > 2\pi$:
a) u_k has exactly two vortices, one of degree 1 near ∂A, the other one of degree -1 near $\partial\omega_0$;
b) any a.e. limit u of the u_κ's is vortexless. More precisely, it is a constant ;
c) up to subsequences, there are exactly two concentration points, one on $\partial\Omega$, the other one on $\partial\omega_0$. All the other points of $\overline{A}$ are regular.

Note the contrast in between the subcritical/critical and the supercritical domains, when we consider the behavior of solutions for large values of κ : in the first case, minimizers do not vanish, by Theorem 2. With more work, one can prove that quasi-minimizers do not vanish neither. In the supercritical case, however, quasi-minimizers do have zeroes (and minimizers presumably cease to exist). In a different context (S^2-valued harmonic maps with Dirichlet boundary conditions in a circular annulus of radii R and ρ), the existence of a critical value of R/ρ determining a qualitative change in the behavior of minimizers was established in [8]. Similar split in behavior was described in physical context, see, e.g., [22].

Finally, we discuss uniqueness of minimizers. Note that, if the minimization problem (1.1)-(1.3) has a solution, then it has infinitely many, since whenever u_κ is a minimizer, so is αu_κ, $\forall\ \alpha \in S^1$. Thus we can, at best, prove uniqueness modulo S^1. In Section 10, we adapt the methods developed in [21] and [31] and establish

Theorem 4 *Assume that A is subcritical or critical. Then, for large κ, the minimizers of (1.1)-(1.3) are unique modulo multiplication with constants of modulus* 1.

Nevertheless, our analysis does not include the result of [27] which asserts that, if A is a circular annulus of sufficiently small capacity, then the minimizers of (1.1)-(1.3) are unique **for all** κ. In this context, we mention the following natural question concerning circular annuli

Open Problem. Let $A = \{z\ ;\ \rho < |z| < R\}$. Assume that A is subcritical or critical. Is it true that, **for all** κ, the minimizers of (1.1)-(1.3) are unique modulo S^1 ?

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Appendices

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2 Properties of the class $\mathcal{K}$ and of the degree

2.1 The class K

We discuss here some properties of the class $\mathcal{K}$ defined in the Introduction. For later use, it will be of interest to consider, more generally, the class

$$K = K_{D, d_0, \ldots, d_k} = \{u \in H^1(A; S^1);\ \deg(u, \partial\Omega) = D,\ \deg(u, \partial\omega_j) = d_j, j = 0, \ldots, k\}. \tag{2.1}$$

The properties of K we present below are well-known to experts. However, since part of these results are not published yet, we will also present some proofs in Appendix A. The main references for this section are [10], [15] and [20].

Lemma 2.1. ([15]) *Let $u \in H^1(A; C)$. Then*

$$\int_A \text{Jac } u = \frac{1}{2} \int_{\partial A} u \wedge \frac{\partial u}{\partial \tau}. \tag{2.2}$$

Lemma 2.2. **([15])** *We have*

$$K \neq \emptyset \Longleftrightarrow D + \sum_{j=0}^{k} d_j = 0. \tag{2.3}$$

Lemma 2.3. **([15])** *Assume the compatibility condition*

$$D + \sum_{j=0}^{k} d_j = 0 \tag{2.4}$$

satisfied. Let $v \in K$ be fixed. Then :
a) we have

$$K = \{ve^{\imath\varphi} \ ; \ \varphi \in H^1(A; I\!R)\}; \tag{2.5}$$

b) K is closed with respect to weak H^1 convergence.

We now recall the main result in [10] concerning the class K :

Lemma 2.4. **([10])** *Assume (2.4) satisfied. Let*

$$I = I_{D, d_0, \ldots, d_k} = \mathrm{Min}\,\Big\{\frac{1}{2}\int_A |\nabla u|^2 \ ; \ u \in K\Big\}. \tag{2.6}$$

Then I is attained. Moreover :
a) the minimizer is unique up to a phase shift, i.e., if u, v are two minimizers of (2.6), then $u = \alpha v$ for some $\alpha \in S^1$;
b) any minimizer is smooth ;
c) we have

$$I = \frac{1}{2}\int_A |\nabla \eta|^2. \tag{2.7}$$

Here, η is smooth and it is the only minimizer of

$$\mathrm{Min}\,\Big\{\frac{1}{2}\int_A |\nabla \zeta|^2 + 2\pi \sum_{j=0}^{k} d_j \zeta_{|\partial\omega_j}; \ \zeta \in L\Big\}, \tag{2.8}$$

where

$$L = \{\zeta \in H^1(A; I\!R); \ \zeta = 0 \text{ on } \partial\Omega, \ \zeta = \text{const. on each } \partial\omega_j, j = 0, \ldots, k\}\,; \tag{2.9}$$

d) if u is any minimizer of (2.6) and if η is as above, then

$$u \wedge \nabla u = \begin{pmatrix} -\partial\eta/\partial y \\ \partial\eta/\partial x \end{pmatrix}; \tag{2.10}$$

e) if u is any minimizer of (2.6), we may write locally (i.e., in simply connected sub domains of A) $u = e^{\imath\varphi}$, with φ smooth. The quantity $X = \nabla\varphi$ is globally defined, it can be computed as $X = u \wedge \nabla u$ and is the only solution of

$$\begin{cases} div\ X = 0 & in\ A \\ X \cdot \nu = 0 & on\ \partial A \\ \int\limits_{\partial\omega_j} X \cdot \tau = 2\pi d_j, & j = 0, \dots, k. \end{cases} \tag{2.11}$$

Moreover, we have

$$|X| = |\nabla u|; \tag{2.12}$$

f) the function η defined above is the only solution of

$$\begin{cases} \Delta\eta = 0 & in\ A \\ \eta = 0 & on\ \partial\Omega \\ \eta = C_j & on\ \partial\omega_j,\ j = 0, \dots k \\ \int\limits_{\partial\omega_j} \dfrac{\partial\eta}{\partial\nu} = 2\pi d_j, & j = 0, \dots, k \end{cases} . \tag{2.13}$$

(Here, the constants C_j are a priori unknown and part of the problem.)

2.2 The class $\mathcal{K}$

¿From now on, we specialize to the class $\mathcal{K}$, i.e., we will always assume in what follows that

$$D = 1,\ d_0 = -1,\ d_j = 0,\ j = 1, \dots, k. \tag{2.14}$$

Note that $\mathcal{K}$ satisfies the compatibility condition (2.4), and thus Lemma 2.4 applies to $\mathcal{K}$. In agreement with the notation used in the Introduction, we will write I_0 instead of $I_{1,\,-1,\,0,\,\dots,\,0}$.

The following properties of the function η introduced above will be useful later :

Lemma 2.5. *Assume (2.14) satisfied. Then :*
a) $0 > C_j > C_0$, $j = 1, \dots, k$ and $0 > \eta > C_0$ in A ;
b) if $t \in (C_0, 0)$ is not a critical value of η, then the level set $\{\eta = t\}$ consists of a single simple curve which encloses $\partial\omega_0$ and

$$\int\limits_{\{\eta = t\}} |\nabla\eta| = 2\pi; \tag{2.15}$$

c) I_0 and C_0 are related by

$$C_0 = -\frac{I_0}{\pi}. \tag{2.16}$$

In case of a circular annulus, we have explicit formulae for u, η, I_0 and C_0 :

Lemma 2.6. *Assume that $A = \{z\,;\ \rho < |z| < R\}$, i.e., $\Omega = \{z\,;\ |z| < R\}$ and $\omega_0 = \{z\,;\ |z| < \rho\}$. Then :*
a) all the minimizers of (1.14)-(1.15) are of the form $u(z) = \alpha \dfrac{z}{|z|}$ for some $\alpha \in S^1$;
b) $\eta(z) = \ln|z| - \ln R$;
c) $C_0 = \ln\rho - \ln R$;
d) $I_0 = \pi \ln \dfrac{R}{\rho}$.

Corollary 2.1. *When A is a circular domain, $A = \{z\,;\ \rho < |z| < R\}$, the subcritical case corresponds to $R/\rho < e^2$, the critical case to $R/\rho = e^2$ and the supercritical case to $R/\rho > e^2$.*

We now turn to domains of the form $A = \Omega \setminus \omega_0$. As we will see, in this case I_0 and η are related to the H^1-capacity of A. We recall the definition of the H^1-capacity of a hole in a 2D domain (see, e.g., [32]) :

Definition 2.1. *Let ω_0, Ω be smooth bounded simply connected domains in $I\!R^2$ such that $\overline{\omega_0} \subset \Omega$. Set $A = \Omega \setminus \omega_0$. Then*

$$\mathrm{cap}(A) = \mathrm{Min}\ \{\int_A |\nabla v|^2\ ;\ v \in H^1(A), v = 0 \text{ on } \partial\Omega, v = 1 \text{ on } \partial\omega_0\}. \tag{2.17}$$

Lemma 2.7. *Assume $A = \Omega \setminus \omega_0$. Then*

$$I_0 = \frac{2\pi^2}{\mathrm{cap}(A)}. \tag{2.18}$$

Finally, we consider a general perforated domain $A = \Omega \setminus \cup_{j=0}^k \omega_j$. In this case, we introduce the following analogue of the H^1-capacity

Definition 2.2. *The generalized H^1-capacity of the domain $A = \Omega \setminus \cup_{j=0}^k \omega_j$ is*

$$\mathrm{cap}(A) = \mathrm{Min}\ \{\int_A |\nabla v|^2\ ;\ v \in H^1(A), v = 0 \text{ on } \partial\Omega, v = 1 \text{ on } \partial\omega_0, v = D_j \text{ on } \partial\omega_j, j = 1, \ldots, k\}. \tag{2.19}$$

In (2.19), the minimum is taken among all the v's and all the constants D_j. Note that the hole ω_0 plays a special role. It is easy to see that the minimum is attained in (2.19), that the minimizer

v of (2.19) is unique and satisfies

$$\begin{cases} \Delta v = 0 & \text{in } A \\ v = 0 & \text{on } \partial\Omega \\ v = 1 & \text{on } \partial\omega_0 \\ v = D_j & \text{on } \partial\omega_j,\ j = 1, \dots k\ \cdot \\ \displaystyle\int_{\partial\omega_j} \frac{\partial v}{\partial \nu} = 0, & j = 1, \dots, k \end{cases} \tag{2.20}$$

The proof of Lemma 2.7 (presented in Appendix A) combined with (2.20) yields immediately

Lemma 2.8. *Assume $A = \Omega \setminus \cup_{j=0}^{k} \omega_j$. Then*

$$I_0 = \frac{2\pi^2}{\mathrm{cap}(A)}. \tag{2.21}$$

2.3 Symmetric domains

We end this section by considering symmetric domains. In this case, we prove that there are minimizers u_0 of I_0 that inherit the symmetry properties of the domain. Since the hole ω_0 plays a distinguished role, we have to start by providing a good notion of symmetric domains.

Definition 2.3. *Let $\mathcal{O}$ be an isometry of the plane. The domain $A = \Omega \setminus \cup_{j=0}^{k} \omega_j$ is symmetric with respect to $\mathcal{O}$ if*

$$\mathcal{O}(A) = A \quad \text{and} \quad \mathcal{O}(\omega_0) = \omega_0. \tag{2.22}$$

Note that, if A is not a circular annulus, we may assume that $\mathcal{O}$ is either an orthogonal symmetry with respect to a line, or a $\mathcal{O}$ a rotation of angle $2\pi/n$, $n \geq 2$. Indeed, if $\mathcal{O}$ is a rotation of angle $2\pi\theta$, with $\theta \notin \mathbb{Q}$, then A has to be a circular annulus. On the other hand, if $\mathcal{O}$ is a rotation of angle $2\pi m/n$, with $(m, n) = 1$, then an appropriate iteration of $\mathcal{O}$ is of a rotation of angle $2\pi/n$ and invariates A.

Lemma 2.9. *Assume that A is $\mathcal{O}$-symmetric. Then there is a minimizer u_0 of I_0 such that*

$$u_0(\mathcal{O}(z)) = \mathcal{O}(u_0(z)), \quad \forall\ z \in A. \tag{2.23}$$

3 The geometrical interpretation of the capacity

As we will see below, the capacity $\mathrm{cap}(A)$ is related to conformal representations. We recall some well known facts about conformal representations of multiply connected domains. We follow essentially [1]. To start with, consider the case $A = \Omega \setminus \omega_0$. Recall that, in this case, A can be conformally mapped into a circular annulus $\{z\,; \rho < |z| < R\}$ (see, e.g., [1]). Moreover, the ratio

R/ρ is uniquely determined by A. Indeed, circular annuli are conformally rigid, i.e., two annuli $\{z\,;\rho_1 < |z| < R_1\}$ and $\{z\,;\rho_2 < |z| < R_2\}$ are conformally equivalent if and only if $R_1/\rho_1 = R_2/\rho_2$ ([1]). It turns out that the ratio R/ρ is related to $\operatorname{cap}(A)$ in a simple way and that the maps defined in Lemma 2.4, namely u and η, provide an explicit representation of A into $\{z\,;\rho < |z| < R\}$.

Definition 3.1. *If u is a minimizer of (1.14)-(1.15) and η is the map defined in Lemma 2.4, let*

$$f = f_{A,\,u} = e^{\eta}u. \tag{3.1}$$

It is clear from Lemma 2.4 that f is holomorphic.

Part of the following result is proved in [1] :

Lemma 3.1. *Assume that $A = \Omega \setminus \omega_0$. Then :*
a) if ρ, R are such that A can be conformally represented into $\{z\,;\rho < |z| < R\}$, then

$$\frac{R}{\rho} = \exp\Big(\frac{2\pi}{\operatorname{cap}(A)}\Big); \tag{3.2}$$

b) the map f is a conformal representation of A into the circular annulus

$$\mathcal{C} = \Big\{z\,;\exp\Big(-\frac{2\pi}{\operatorname{cap}(A)}\Big) < |z| < 1\Big\}. \tag{3.3}$$

c) f extends to a C^1-diffeomorphism from $\overline{A}$ into $\overline{\mathcal{C}}$ such that $f(\partial\Omega) = \{z\,;|z| = 1\}$ and $f(\partial\omega_0) = \Big\{z\,;|z| = \exp\Big(-\frac{2\pi}{\operatorname{cap}(A)}\Big)\Big\}$. Moreover, f preserves the natural orientation of simple curves.

Proof : Part a) follows from b) and the conformal rigidity of circular annuli. Part b) is proved in [1] except that the explicit formula for the small radius of $\mathcal{C}$ obtained in [1] is $\rho = e^{C_0}$. But this ρ is exactly the one given in b), thanks to Lemmas 2.6 and 2.8. We now turn to the proof of c). On the one hand, it is clear from the definition of f that f is smooth up to the boundary. Since $|f| = e^{C_0}$ on $\partial\omega_0$, we have $f(\partial\omega_0) \subset \Big\{z\,;|z| = \exp\Big(-\frac{2\pi}{\operatorname{cap}(A)}\Big)\Big\}$; similarly, we have $f(\partial\Omega) \subset \{z\,;|z| = 1\}$. On $\partial\omega_0$, we have

$$\frac{\partial\varphi}{\partial\tau} = \frac{\partial\eta}{\partial\nu} < 0 \text{ and } \int_{\partial\omega_0}\frac{\partial\varphi}{\partial\tau} = \int_{\partial\omega_0}\frac{\partial\eta}{\partial\nu} = -2\pi.. \tag{3.4}$$

Thus (with the natural orientations), f is an orientation preserving diffeomorphism from $\partial\omega_0$ into $\Big\{z\,;|z| = \exp\Big(-\frac{2\pi}{\operatorname{cap}(A)}\Big)\Big\}$. Similar assertion holds for $\partial\Omega$. Finally, f preserves the natural orientation of any simple curve in $\overline{A}$, since it does so for $\partial\omega_0$.

We now turn to a general $A = \Omega \setminus \cup_{j=0}^{k}\omega_j$. Recall the following

Definition 3.2. ([1]) *A* **canonical slit region** *is a set $\mathcal{E}$ of the form*

$$\mathcal{E} = \{z\,;\rho < |z| < R\} \setminus \cup_{j=1}^{k} \Gamma_j. \tag{3.5}$$

Here, each Γ_j is a closed circular arc properly contained into some circle $\{z\,;|z| = R_j\}$, $\rho < R_j < R$, and these arcs are mutually disjoint.

We quote the following version of Lemma 3.1, essentially proved in [1] :

Lemma 3.2. *Assume $A = \Omega \setminus \cup_{j=0}^{k} \omega_j$. Then :*
a) the map $f_{A,\,u}$ is a conformal representation of A into a canonical slit region $\mathcal{C}$ of radii $R = 1$, $\rho = \exp\left(-\dfrac{2\pi}{\mathrm{cap}(A)}\right)$;
b) f extends to a C^1-diffeomorphism from $A \cup \partial\Omega \cup \partial\omega_0$ into $\mathcal{C} \cup \{z\,;|z| = 1\} \cup \left\{z\,;|z| = \exp\left(-\dfrac{2\pi}{\mathrm{cap}(A)}\right)\right\}$ such that $f(\partial\Omega) = \{z\,;|z| = 1\}$ and $f(\partial\omega_0) = \left\{z\,;|z| = \exp\left(-\dfrac{2\pi}{\mathrm{cap}(A)}\right)\right\}$. Moreover, f preserves the natural orientation of simple curves ;
c) $\Gamma_j = f(\partial\omega_j)$, $j = 1, \ldots, k$;
d) if A can be represented into a canonical slit region $\mathcal{F}$ of radii $\rho < R$ through some conformal mapping h such that $|h|_{|\partial\omega_0} < |h|_{|\partial\Omega}$, then there are some $\alpha \in C \setminus \{0\}$, $\beta \in S^1$ such that $\mathcal{F} = \alpha\mathcal{C}$ and $h = \alpha\beta f$. In particular,

$$\frac{R}{\rho} = \exp\left(\frac{2\pi}{\mathrm{cap}(A)}\right). \tag{3.6}$$

4 Properties of the class $\mathcal{J}$

4.1 On Example 1

We begin this section by discussing in detail the Example 1 mentioned in the Introduction. We will prove the following slightly more general fact

Lemma 4.1. *Let U be a smooth bounded simply connected domain in C. Set, for $\kappa > 0$,*

$$n_\kappa = \mathrm{Inf}\left\{\frac{1}{2}\int_U |\nabla v|^2 + \frac{\kappa^2}{4}\int_U (1-|v|^2)^2\ ;\ v \in H^1(A; C),\ |v| = 1 \text{ a.e. on } \partial U,\ \deg\,(v, \partial U) = 1\right\}. \tag{4.1}$$

Then :
a) $n_\kappa = \pi$;
b) n_κ is never attained.

Proof : Let v be a testing map in (4.1). Since $|\nabla v|^2 \geq 2|\text{Jac } v|$ pointwise, we find that

$$\frac{1}{2}\int_U |\nabla v|^2 \geq \int_U |\text{Jac } v| \geq \int_U \text{ Jac } v = \frac{1}{2}\int_{\partial U} v \wedge \frac{\partial v}{\partial \tau} = \pi \text{ deg } (v, \partial U) = \pi, \tag{4.2}$$

by Lemma 2.1 and the degree formula (1.5). Thus $n_\kappa \geq \pi$. We claim that there is no testing map v such that

$$\frac{1}{2}\int_U |\nabla v|^2 + \frac{\kappa^2}{4}\int_U (1-|v|^2)^2 = \pi. \tag{4.3}$$

Indeed, by (4.2) this would require $|v| = 1$ a.e., so that $v \in H^1(U; S^1)$. However, in this case Lemma 2.2 implies that deg $(v, \partial U) = 0$, which is the desired contradiction. We complete the proof by showing that $n_\kappa = \pi$. Since U is smooth, U can be conformally represented into the unit disc $I\!\!D$ through a map w which extends as a C^1-diffeomorphism from $\overline{U}$ into $\overline{I\!\!D}$. Moreover, since U is bounded, w preserves the natural orientations, so that we have deg $(w, \partial U) = 1$; thus w is in the class of testing maps. Consider now, for $a \in I\!\!D$, $\alpha \in S^1$, the Moebius conformal representations of $\overline{I\!\!D}$ into itself,

$$u_{a,\alpha}(z) = \alpha\frac{z-a}{1-\overline{a}z}, \quad \forall\, z \in \overline{I\!\!D}, \tag{4.4}$$

and let $u_a = u_{a,1}$. Set $v_a = u_a \circ w$, which is again a testing map, since u_a preserves the orientation of S^1. By using repeatedly conformality, we have

$$\frac{1}{2}\int_U |\nabla v_a|^2 = \int_U |\text{ Jac } v_a| = \int_U \text{ Jac } v_a = \text{area}(v_a(U)) = \pi. \tag{4.5}$$

On the other hand, if we consider a real, $a \in (0,1)$, we claim that

$$\lim_{a \nearrow 1} \frac{\kappa^2}{4}\int_U (1-|v_a|^2)^2 = \lim_{a \nearrow 1} \frac{\kappa^2}{4}\int_{I\!\!D} (1-|u_a|^2)^2 \text{ Jac } w = 0. \tag{4.6}$$

Indeed, the last equality in (4.6) follows by dominated convergence, using the fact that, for each fixed $z \in I\!\!D$, we have $u_a(z) \to -1$ as $a \nearrow 1$.

Remark 4.1. Here is another similar example :
Let $A = \Omega \setminus \omega_0$ and consider the class

$$J = \{u \in H^1(A; \mathbb{C})\,;\, |u| = 1 \text{ on } \partial A,\, \deg(u, \partial\Omega) = 1,\, \deg(u, \partial\omega_0) = 0\}.$$

Then one may prove that, for each $\kappa \geq 0$, we have

$$\text{Inf}\{E_\kappa(u)\,;\, u \in J\} = \pi,$$

and that this infimum is never attained.

4.2 The class J

We now turn to the study of the class $\mathcal{J}$. As in Section 2, we will consider, more generally, the class

$$J = J_{D,\, d_0, \ldots, d_k}$$
$$= \{u \in H^1(A; \mathbb{C}); |u| = 1 \text{ a.e. on } \partial A,\ \deg(u, \partial\Omega) = D,\ \deg(u, \partial\omega_j) = d_j, j = 0, \ldots, k\}. \tag{4.7}$$

In contrast with Lemma 2.2 and Lemma 2.3 b), we have

Lemma 4.2. *The class J is always nonempty and never closed with respect to weak H^1 convergence.*

Proof : Fix $a_j \in \omega_j$, $j = 0, \ldots, k$ and $a \in A$ and let

$$v(z) = \prod_{j=0}^{k} \left(\frac{z - a_j}{|z - a_j|}\right)^{-d_j} \left(\frac{z - a}{|z - a|}\right)^{D + \sum_{j=0}^{k} d_j}. \tag{4.8}$$

Let $g = v_{|\partial A}$. Then any smooth extension of g to A is in J.
In order to prove the second property, let v be any smooth map in $J_{D-1,\, d_0, \ldots, d_k}$. Let w be a conformal representation of $\overline{\Omega}$ into $\overline{I\!D}$ and let u_a be the map defined by (4.4). Set $v_a = u_a \circ w : \overline{\Omega} \to \overline{I\!D}$. We are going to modify v_a in order to obtain a map having modulus 1 on ∂A ; v_a does not have this property, since we only have $|v_a| = 1$ on $\partial\Omega$. We start by estimating $|v_a|$ on $\cup_{j=0}^{k} \partial\omega_j$. Let $K = w(\cup_{j=0}^{k} \overline{\omega_j})$, which is a compact in $I\!D$. It is easy to see that there is some $C > 0$ such that

$$|u_a(z)| \geq 1 - C(1 - a), \quad \forall\ z \in K,\ \forall\ a \in (1/2, 1), \tag{4.9}$$

and thus

$$|v_a(z)| \geq 1 - C(1 - a), \quad \forall\ z \in \cup_{j=0}^{k} \partial\omega_j,\ \forall\ a \in (1/2, 1). \tag{4.10}$$

We define now the following family of maps $\Phi_t : \mathbb{C} \to \overline{I\!D}$, $0 < t < 1/4$:

$$\Phi_t(z) = \begin{cases} z, & \text{if } |z| \leq 1 - 2t \\ \dfrac{z}{|z|}, & \text{if } |z| \geq 1 - t \\ \left(2 - \dfrac{1 - 2t}{|z|}\right) z, & \text{if } 1 - 2t \leq |z| \leq 1 - t \end{cases},$$

which clearly satisfies

$$\Phi_t(z) = \frac{z}{|z|}, \text{ if } |z| \geq 1 - t, \quad |\nabla\Phi_t - \nabla \mathrm{id}| \leq Ct \text{ if } |z| \leq 1, \quad |\nabla\Phi_t| \leq C, \tag{4.11}$$

for some constant C independent of t. Let

$$w_a(z) = v(z)\Phi_{\sqrt{1-a}} \circ v_a(z), \quad \forall\ z \in A, \forall\ a \in (1/2, 1).$$

By (4.10) and (4.11), $w_a \in J$ provided a is sufficiently close to 1. Moreover, we have $w_a \to -v \notin J$ a.e. as $a \nearrow 1$. This proves the lack of weak closedness of J provided we establish that (w_a) is bounded in $H^1(A)$. This is clear since on the one hand we have $|w_a| \le |v| \le C$ and on the other hand we have

$$\int_A |\nabla w_a|^2 \le 2\int_A (|v|^2|\nabla(\Phi_{\sqrt{1-a}} \circ v_a)|^2 + |\Phi_{\sqrt{1-a}} \circ v_a|^2|\nabla v|^2) \le C\int_A (|\nabla v_a|^2 + |\nabla v|^2) \le C, \quad (4.12)$$

by (4.5).

We next establish a lower bound for maps in J that will be useful later.

Lemma 4.3. *Let $u \in J$. Then*

$$\frac{1}{2}\int_A |\nabla u|^2 \ge \pi\Big|D + \sum_{j=0}^{k} d_j\Big|. \quad (4.13)$$

Proof : We have, by Lemma 2.1 and the degree formula (1.5),

$$\frac{1}{2}\int_A |\nabla u|^2 \ge \int_A |\mathrm{Jac}\ u| \ge \Big|\int_A \ \mathrm{Jac}\ u\Big| = \frac{1}{2}\Big|\int_{\partial A} u \wedge \frac{\partial u}{\partial \tau}\Big| = \pi\Big|D + \sum_{j=0}^{k} d_j\Big|. \quad (4.14)$$

4.3 Smoothness of critical points

We state here the following regularity result, whose proof is presented in Appendix C

Lemma 4.4. *Let $v_\kappa \in \mathcal{J}$ be a critical point of the Ginzburg-Landau energy E_κ with respect to $\mathcal{J}$. Then :*
a) $v_\kappa \in C^\infty(\overline{A})$. In particular, near ∂A we may locally write $v_\kappa = \rho e^{\imath\psi}$;
b) v_κ satisfies the system

$$\begin{cases} -\Delta v_\kappa &= \kappa^2 v_\kappa(1-|v_\kappa|^2) & \text{in } A \\ \rho &= 1 & \text{on } \partial A \\ \dfrac{\partial \psi}{\partial \nu} &= 0 & \text{on } \partial A \end{cases} ; \quad (4.15)$$

c) $|v_\kappa| \le 1$ in A.

Remark 4.2. Near ∂A, the (local) phase ψ of v_κ is not unique. However, $\nabla\psi$ is uniquely determined and may be computed as $\dfrac{v_\kappa}{|v_\kappa|} \wedge \nabla\Big(\dfrac{v_\kappa}{|v_\kappa|}\Big)$. Thus, the last equation in (4.15) is meaningful.

Remark 4.3. For later use, we mention that, if in the set $\{z\ ;\ v_\kappa \neq 0\}$, we write locally (i.e., on simply connected domains V), $v_\kappa = \rho_\kappa e^{\imath\psi_\kappa} = \rho e^{\imath\psi}$, then ρ and ψ satisfy

$$\begin{cases} -\Delta\rho &= \kappa^2\rho(1-\rho^2) - \rho|\nabla\psi|^2 \quad \text{in } V \\ \quad \rho &= 1 \qquad\qquad\qquad\qquad\quad\ \text{on } \partial A \cap V \end{cases} \tag{4.16}$$

and respectively

$$\begin{cases} -\mathrm{div}(\rho^2\nabla\psi) &= 0 \quad \text{in } V \\ \qquad \nu\cdot\nabla\psi &= 0 \quad \text{on } \partial A \cap V \end{cases}. \tag{4.17}$$

5 On the existence of minimizers

5.1 A sufficient condition for the existence of minimizers

This part is devoted to the proof of the following

Proposition 5.1. *Assume that $m_\kappa < 2\pi$. Then m_κ is attained.*

Proof : Let (u_n) be a minimizing sequence for E_κ in $\mathcal{J}$ and let u be such that, up to a subsequence $u_n \rightharpoonup u$ weakly in $H^1(A)$. Set $g_n = \mathrm{tr}_{\partial A} u_n$, $g = \mathrm{tr}_{\partial A} u$, so that $g_n \rightharpoonup g$ weakly in $H^{1/2}(\partial A)$. Since $H^{1/2}(\partial A)$ is compactly embedded into $L^2(\partial A)$, we have $g_n \to g$ in $L^2(\partial A)$. In particular, up to some further subsequence, we may assume that $g_n \to g$ a.e. Thus $|g| = 1$ a.e. on ∂A, and therefore $u \in J_{D,\,d_0,\,\dots,\,d_k}$ for some integers $D, d_0, \dots, d_k$. By the Fatou lemma, we have

$$E_\kappa(u) \le \lim_{n\to\infty} E_\kappa(u_n) = m_\kappa. \tag{5.1}$$

Therefore, it suffices to prove that $D = 1$, $d_0 = -1$, $d_j = 0$, $j = 1, \dots, k$, i.e., that $u \in \mathcal{J}$.
We have

$$\frac{1}{2}\int_A |\nabla u_n|^2 = \frac{1}{2}\int_A |\nabla((u_n - u) + u)|^2 = \frac{1}{2}\int_A |\nabla(u_n - u)|^2 + \frac{1}{2}\int_A |\nabla u|^2 + \int_A \nabla(u_n - u)\cdot\nabla u, \tag{5.2}$$

which implies

$$\frac{1}{2}\int_A |\nabla u_n|^2 = \frac{1}{2}\int_A |\nabla(u_n - u)|^2 + \frac{1}{2}\int_A |\nabla u|^2 + o(1) \quad \text{as } n\to\infty, \tag{5.3}$$

since $u_n - u \rightharpoonup 0$ weakly in $H^1(A)$.
Let v_n, v be the harmonic extensions of g_n, g respectively. Using the fact that $g_n \rightharpoonup g$ weakly in $H^{1/2}(\partial A)$, we find that $v_n \to v$ in $C^1_{\mathrm{loc}}(A)$, by standard elliptic estimates ([26]). Consider smooth bounded disjoint neighborhoods of the ω_j's, $U_0, \dots, U_k$, such that

$$\overline{\omega_j} \subset U_j,\ j = 0, \dots, k,\ \overline{U_j} \subset \Omega,\ j = 0, \dots, k,\ \overline{U_j} \cap \overline{U_l} = \emptyset,\ j \neq l, \tag{5.4}$$

and let U be a smooth domain such that

$$\cup_{j=0}^k \overline{U_j} \subset U \subset \overline{U} \subset \Omega. \tag{5.5}$$

Since

$$\frac{1}{2}\int_A |\nabla(u_n - u)|^2 \geq \frac{1}{2}\int_A |\nabla(v_n - v)|^2 \geq \int_{\Omega\setminus U} |\nabla(v_n - v)|^2 + \sum_{j=0}^k \int_{U_j\setminus\omega_j} |\nabla(v_n - v)|^2, \tag{5.6}$$

we find, using the pointwise inequality $|\nabla(v_n - v)|^2 \geq 2|\text{Jac }(v_n - v)|$ and Lemma 2.1, that

$$\frac{1}{2}\int_A |\nabla(u_n - u)|^2 \geq \frac{1}{2}\Big|\int_{\partial(\Omega\setminus U)} (v_n - v)\wedge\frac{\partial(v_n - v)}{\partial\tau}\Big| + \frac{1}{2}\sum_{j=0}^k\Big|\int_{\partial(U_j\setminus\omega_j)} (v_n - v)\wedge\frac{\partial(v_n - v)}{\partial\tau}\Big|. \tag{5.7}$$

Recalling that $v_n \to v$ in $C^1_{\text{loc}}(A)$, we obtain

$$\int_{\partial(\Omega\setminus U)} (v_n - v)\wedge\frac{\partial(v_n - v)}{\partial\tau} = \int_{\partial\Omega} (v_n - v)\wedge\frac{\partial(v_n - v)}{\partial\tau} + o(1), \quad \text{as } n\to\infty \tag{5.8}$$

and

$$\int_{\partial(U_j\setminus\omega_j)} (v_n - v)\wedge\frac{\partial(v_n - v)}{\partial\tau} = \int_{\partial\omega_j} (v_n - v)\wedge\frac{\partial(v_n - v)}{\partial\tau} + o(1),\ j = 1,\dots,k, \quad \text{as } n\to\infty. \tag{5.9}$$

Since u_n and v_n (respectively u and v) agree on ∂A we have, by the degree formula (1.5),

$$\int_{\partial\Omega} v_n\wedge\frac{\partial v_n}{\partial\tau} = 2\pi, \int_{\partial\Omega} v\wedge\frac{\partial v}{\partial\tau} = 2\pi D, \tag{5.10}$$

$$\int_{\partial\omega_0} v_n\wedge\frac{\partial v_n}{\partial\tau} = -2\pi, \int_{\partial\omega_0} v\wedge\frac{\partial v}{\partial\tau} = 2\pi d_0 \tag{5.11}$$

and

$$\int_{\partial\omega_j} v_n\wedge\frac{\partial v_n}{\partial\tau} = 0, \int_{\partial\omega_j} v\wedge\frac{\partial v}{\partial\tau} = 2\pi d_j,\ j = 1,\dots,k. \tag{5.12}$$

Using the weak convergence of traces in $H^{1/2}$, we also have, for any component Γ of ∂A, that

$$\int_\Gamma v_n\wedge\frac{\partial v}{\partial\tau} = \int_\Gamma v\wedge\frac{\partial v}{\partial\tau} + o(1), \quad \text{as } n\to\infty. \tag{5.13}$$

Finally, we use the fact that

$$\int\limits_{\Gamma} v \wedge \frac{\partial v_n}{\partial \tau} = -\int\limits_{\Gamma} \frac{\partial v}{\partial \tau} \wedge v_n = \int\limits_{\Gamma} v_n \wedge \frac{\partial v}{\partial \tau}. \tag{5.14}$$

The above equality is clear, using integration by parts, when both v_n and v are smooth. The general is obtained through approximation with smooth functions. By combining (5.3) with (5.7)-(5.14), we find

$$2\pi > m_\kappa \geq \liminf \frac{1}{2}\int\limits_{A} |\nabla u_n|^2 \geq \pi\Big(|D-1| + |d_0+1| + \sum_{j=1}^{k} |d_j|\Big) + \frac{1}{2}\int\limits_{A} |\nabla u|^2, \tag{5.15}$$

which proves the Price Lemma (Lemma 1). Using the lower bound provided by Lemma 4.3, we are finally led to

$$2\pi > \pi\Big(|D-1| + |d_0+1| + \sum_{j=1}^{k} |d_j|\Big) + \pi\Big|D + \sum_{j=0}^{k} d_j\Big| \equiv \pi M + \pi N. \tag{5.16}$$

We claim that the right-hand side of (5.16) is $\geq 2\pi$ unless

$$D = 1,\ d_0 = -1,\ d_j = 0,\ j = 1, \dots, k\ ; \tag{5.17}$$

in other words, that (5.16) implies that $u \in \mathcal{J}$.
Indeed, if (5.17) does not hold, then : either exactly one of the equalities in (5.17) is violated, and thus $D + \sum_{j=0}^{k} d_j \neq 0$, and the conclusion is clear, since $M \geq 1$ and $N \geq 1$; or, at least two of the inequalities in (5.17) are false, and then the conclusion is again clear, since $M \geq 2$. Therefore, $u \in \mathcal{J}$ and the proof of the proposition is complete.

If we examine the above proof, we see that it has as a byproduct the following

Corollary 5.1. *Assume that $(u_n) \subset \mathcal{J}$ is such that $u_n \rightharpoonup u$ weakly in $H^1(A)$ to some $u \notin \mathcal{J}$. Then*

$$\liminf_{n\to\infty} \frac{1}{2}\int\limits_{A} |\nabla u_n|^2 \geq max\Big\{2\pi\ ,\ \pi + \frac{1}{2}\int\limits_{A} |\nabla u|^2\Big\} \geq 2\pi. \tag{5.18}$$

Corollary 5.2. *Assume that $I_0 < 2\pi$. Then m_κ is attained for each $\kappa > 0$.*

Proof : Since any minimizer u of (1.14)-(1.15) belongs to $\mathcal{J}$, we have

$$m_\kappa \leq E_\kappa(u) = I_0 < 2\pi. \tag{5.19}$$

Corollary 5.3. *Assume that $I_0 = 2\pi$. Then m_κ is attained for each $\kappa > 0$.*

Proof : Let u be a minimizer of (1.14)-(1.15)and g be the restriction of u to ∂A. Let w attain the minimum of E_κ in the following subclass of $\mathcal{J}$:

$$\mathcal{L} = \{v \in H^1(A; \mathbb{C});\ \mathrm{tr}_{\partial A} v = g\}. \tag{5.20}$$

The minimum is clearly attained, and any minimizer w belongs to $\mathcal{J}$ and satisfies the Ginzburg-Landau equation

$$-\Delta w = \kappa^2 w(1 - |w|^2). \tag{5.21}$$

We claim that u (which belongs to $\mathcal{L}$) is not a minimizer of E_κ in $\mathcal{L}$. Indeed, otherwise we would have, by (5.21) and the fact that $|u| = 1$, that $\Delta u = 0$. Using once again the fact that $|u| = 1$, we find that u is a constant. This contradicts the fact that $u \in \mathcal{K}$. In conclusion,

$$E_\kappa(w) < E_\kappa(u) = I_0 = 2\pi, \tag{5.22}$$

so that $m_\kappa < 2\pi$ and the conclusion follows from Proposition 5.1.

Corollary 5.4. *Assume that $m_{\kappa'} < 2\pi$ for some κ'. Then there is some $\kappa'' > \kappa'$ such that m_κ is attained if $k' < \kappa < \kappa''$.*

Proof : Let $v \in \mathcal{J}$ be such that $E_{\kappa'}(v) < 2\pi$. Then clearly $E_\kappa(v) < 2\pi$ if κ is sufficiently close to κ', and we conclude by applying Proposition 5.1 to any such κ.

5.2 Existence of minimizers for small κ when $A = \Omega \setminus \omega_0$

Throughout this part, we assume that A is an annular domain, i.e., that $A = \Omega \setminus \omega_0$. In this case, we prove that, for small values of κ, there is a minimizer of (1.1)-(1.3). Moreover, we will determine the exact value of m_0, as well as all the minimizers of (1.1)-(1.3). We start with the following

Lemma 5.1. *Assume that $A = \Omega \setminus \omega_0$. Then $m_0 < 2\pi$.*

By combining Lemma 5.1 and Corollary 5.4 we obtain the following

Corollary 5.5. *Assume that $A = \Omega \setminus \omega_0$. Then there is some $\kappa_1 > 0$ such that m_κ is attained for $0 \leq \kappa < \kappa_1$.*

Proof of Lemma 5.1 : We could obtain Lemma 5.1 directly from Proposition 5.2. However, we feel that the argument below, which provides, for a given harmonic map, the harmonic extension of its trace, has its own interest.

Let u be a fixed minimizer of (1.14)-(1.15) and let η be as in Lemma 2.4. Let f be a smooth real function to be determined later and set $u_0 = f(\eta)u$. Then

$$\frac{1}{2}\int_A |\nabla u_0|^2 = \frac{1}{2}\int_A (f'^2(\eta)|\nabla\eta|^2 + f^2(\eta)|\nabla u|^2) = \frac{1}{2}\int_A (f'^2(\eta) + f^2(\eta))|\nabla\eta|^2, \tag{5.23}$$

by Lemma 2.4. By the coarea formula (see, e.g., [25]) and (5.23), we have

$$\frac{1}{2}\int_A |\nabla u_0|^2 = \frac{1}{2}\int_{I\!R}\Big(\int_{\{\eta=t\}} (f'^2(t)+f^2(t))|\nabla\eta|dl\Big)dt = \pi\int_{I\!R}(f'^2(t)+f^2(t))dt\,; \tag{5.24}$$

the last equality in (5.24) follows from Lemma 2.5. Recall that, by Lemma 2.4, η is constant on $\partial\Omega$ and on $\partial\omega_0$; more specifically, by Lemma 2.7 we have $\eta = 0$ on $\partial\Omega$ and $\eta = C_0$ on $\partial\omega_0$, where $C_0 = -\frac{I_0}{\pi}$. Assuming now that $f(0) = f(C_0) = 1$, we obtain $u_0 \in \mathcal{J}$. If $f(0) = f(C_0) = 1$, the right-hand side of (5.24) is minimal for $f(t) = ae^t + be^{-t}$, where a, b satisfy $a + b = 1$ and $ae^{C_0} + be^{-C_0} = 1$. Substituting this f into (5.24) yields

$$m_0 \le \frac{1}{2}\int_A |\nabla u_0|^2 = 2\pi\frac{1-e^{C_0}}{1+e^{C_0}} = 2\pi\frac{1-e^{-I_0/\pi}}{1+e^{-I_0/\pi}} < 2\pi. \tag{5.25}$$

In particular, for a circular annulus $A = \{z\,;\ \rho < |z| < R\}$, we obtain, with the help of Lemma 2.6, that

$$m_0 \le 2\pi\frac{R-\rho}{R+\rho}. \tag{5.26}$$

Remark 5.1. It is easy to see that the map u_0 we constructed above may be also obtained in the following way : let u be a minimizer of (1.14)-(1.15) and let g be its restriction to ∂A. Then u_0 is the harmonic extension of g to A. Recall that, by Lemma 2.4, the minimizers u of I_0 are unique up to a phase shift. Therefore, the maps u_0 constructed above are unique up to a phase shift. We will see below that the above construction is optimal, in the sense that the above maps u_0 are precisely the minimizers of (1.1)-(1.3) for $\kappa = 0$ and that "$\le$" in (5.25) is actually "$=$".

Remark 5.2. If we repeat the above construction when $A = \Omega \setminus (\cup_{j=0}^k \omega_j)$ with $k \ge 1$, in general we obtain $\frac{1}{2}\int_A |\nabla u_0|^2 > 2\pi$; this can be proved by considering appropriate A's. Thus, when $k \ge 1$, we can not derive from the above construction that m_0 is attained.

We next prove that the above construction is optimal.

Proposition 5.2. *Assume that $A = \Omega \setminus \omega_0$. Then :*
a) $m_0 = 2\pi\frac{1-e^{C_0}}{1+e^{C_0}}$;
b) the minimizers of (1.1)-(1.3) for $\kappa = 0$ are precisely the maps $u_0 = f(\eta)u$, where $f(t) = ae^t + be^{-t}$, $a + b = 1$ and $ae^{C_0} + be^{-C_0} = 1$ and u is a minimizer of (1.14)-(1.15).

Proof : We start with the case where A is a circular annulus, $A = \{z\,;\ \rho < |z| < R\}$. Recall that, by Lemma 2.6, in this case we have $u(z) = \alpha\frac{z}{|z|}$. By Remark 5.1, u_0 is the harmonic extension to

A of the restriction of u to ∂A. Therefore, in polar coordinates, we have

$$u = \alpha \frac{r^2 + R\rho}{r(R+\rho)} e^{\imath\theta}. \tag{5.27}$$

By Lemma D.3, the maps given by (5.27) are precisely the minimizers of (1.1)-(1.3) for $\kappa = 0$ and a) holds for this A. In particular, in this case the minimizers of (1.1)-(1.3) for $\kappa = 0$ are unique up to a phase shift.

We now turn to a general $A = \Omega \setminus \omega_0$. Recall that, by Lemma 3.1 c), there is a conformal representation F from A into $\mathcal{C}$ that extends to a C^1 orientation preserving diffeomorphism from $\overline{A}$ into $\overline{\mathcal{C}}$ that preserves the natural orientations of curves ; here, $\mathcal{C} = \{z\, ;\ \rho < |z| < R\}$ and $\dfrac{R}{\rho} = e^{I_0/\pi}$. Thus, with obvious notations, we find that

$$\mathcal{J}(A) \ni v \mapsto v \circ F^{-1} \in \mathcal{J}(\mathcal{C}) \tag{5.28}$$

is a bijection. Moreover, since F is a conformal representation, we have

$$\frac{1}{2}\int_A |\nabla v|^2 = \frac{1}{2}\int_{\mathcal{C}} |\nabla (v \circ F^{-1})|^2. \tag{5.29}$$

In particular, using obvious notations, we find that

$$m_0(A) = m_0(\mathcal{C}) = 2\pi \frac{R-\rho}{R+\rho} == 2\pi \frac{1-(R/\rho)^{-1}}{1+(R/\rho)^{-1}} = 2\pi \frac{1-e^{-I_0/\pi}}{1+e^{-I_0/\pi}}, \tag{5.30}$$

the last equality following from Lemma 3.1.

Therefore, the maps constructed in Lemma 5.1 are, in A, minimizers of (1.1)-(1.3) for $\kappa = 0$. Moreover, since in $\mathcal{C}$ the minimizers of (1.1)-(1.3) for $\kappa = 0$ are determined up to a phase shift and so are the maps constructed in Lemma 5.1, it follows that these maps are all the minimizers in A of (1.1)-(1.3) for $\kappa = 0$.

6 Bounds for the minimal energy m_κ

6.1 Upper bounds for m_κ

We start by constructing appropriate test functions needed in order to derive sharp upper bounds for m_κ.

Lemma 6.1. *There is a sequence $(u_n) \subset J_{1,0,\ldots,0}$ such that :*
a) $|u_n| \le 1$ in A, $\forall\ n$;
b) $u_n \to -1$ in $C^1_{\mathrm{loc}}(\overline{A} \setminus \partial\Omega)$;
c) $\displaystyle\lim_{n\to\infty} \int_A |\nabla u_n|^2 = 2\pi$.

Proof : We make use of the functions considered in the proof of Lemma 4.1. Let w be a conformal representation of Ω into $I\!\!D$ that extends smoothly up to the boundary, let $(a_n) \subset (0,1)$ be a sequence such that $a_n \to 1$ and set $v_n = u_{a_n} \circ w$. We next correct the map v_n in order to have modulus 1 on the boundary. To this purpose, we start by noting that $|v_n + 1| \leq C(1 - a_n)$ on $\partial A \setminus \partial\Omega$. With Φ_t as in the proof of Lemma 4.2, we set $u_n = \Phi_{\sqrt{1-a_n}} \circ v_n$ in A. Then this u_n belongs to $J_{1,0,\ldots,0}$ for sufficiently large n. Properties a) and b) are clear for this choice of u_n, by construction. Moreover, if we extend u_n (the extension being still denoted by u_n) to $\overline{\Omega}$ by the same formula, properties a) and b) still hold with A replaced by Ω. On the other hand, we clearly have, by the property (4.11) of Φ_t, that

$$|\nabla u_n - \nabla v_n| \leq C\sqrt{1-a_n}|\nabla v_n| \quad \text{in } \Omega. \tag{6.1}$$

Therefore,

$$\lim_{n\to\infty} \int_A |\nabla u_n|^2 = \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 = \lim_{n\to\infty} \int_\Omega |\nabla v_n|^2 = 2\int_\Omega |\text{Jac } v_n| = 2\int_\Omega \text{ Jac } v_n = 2 \text{ area } (I\!\!D) = 2\pi, \tag{6.2}$$

since v_n is a conformal representation of Ω into $I\!\!D$.

Similarly, we have

Lemma 6.2. *There is a sequence $(v_n) \subset J_{0,-1,0,\ldots,0}$ such that :*
a) $|v_n| \leq 1$ in A, $\forall\ n$;
b) $v_n \to -1$ in $C^1_{\text{loc}}(\overline{A} \setminus \partial\omega_0)$;
c) $\lim\limits_{n\to\infty} \int_A |\nabla v_n|^2 = 2\pi$.

Proof : We may assume that $0 \in \omega_0$. Let $g(z) = 1/z$ and $B = g(A)$. Then $B = O \setminus \cup_{j=0}^k U_j$, where O is the domain enclosed by $g(\partial\omega_0)$, U_0 the domain enclosed by $g(\partial\Omega)$, and U_j the domain enclosed by $g(\partial\omega_j)$, $j = 1,\ldots,k$. Construct, in B, a sequence (u_n) as in Lemma 6.1 and let $v_n = \overline{u_n} \circ g = \overline{u_n} \circ g^{-1}$. Clearly, (v_n) has the properties a) and b). As for c), it follows from Lemma 6.1 c) and the fact that

$$\int_A |\nabla v_n|^2 = \int_A |\nabla(\overline{u_n} \circ g)|^2 = \int_B |\nabla \overline{u_n}|^2 = \int_B |\nabla u_n|^2, \tag{6.3}$$

since g is a conformal representation.

Remark 6.1. In Section 11, we will give a geometrical interpretation of the maps u_n and v_n constructed above.

We may now establish the following upper bounds

Proposition 6.1. *We have, for each $\kappa \geq 0$ and each A, the following upper bounds*

$$m_\kappa \leq 2\pi \tag{6.4}$$

and

$$m_\kappa < I_0. \tag{6.5}$$

Proof : Inequality (6.5) follows from the proof of Corollary 5.3, where we establish that, with u a minimizer of (1.14)-(1.15), there is some $w \in \mathcal{J}$ such that $E_\kappa(w) < E_\kappa(u) = I_0$.
For the proof of (6.4), let $w_n = u_n v_n$, where u_n, v_n are given by the two preceding lemmas. By (2.13), we have $w_n \in \mathcal{J}$. On the one hand, we have

$$\lim_{n\to\infty} \int_A (1 - |w_n|^2)^2 = 0, \tag{6.6}$$

by dominated convergence, thanks to a) and b) in Lemma 6.1 and Lemma 6.2.
On the other hand, let U, V be smooth open sets such that

$$U, V \subset A,\ U \cap V = \emptyset,\ \overline{U} \cup \overline{V} = \overline{A},\ \partial\Omega \subset \overline{U},\ \partial\omega_0 \subset \overline{V}.$$

Then, by Lemma 6.1. and Lemma 6.2, we find

$$\int_U |u_n \nabla v_n + v_n \nabla u_n|^2 = \int_U |\nabla u_n|^2 + o(1),\ \int_V |u_n \nabla v_n + v_n \nabla u_n|^2 = \int_V |\nabla v_n|^2 + o(1) \ \text{ as } n \to \infty, \tag{6.7}$$

so that

$$\int_A |\nabla w_n|^2 = \int_A |u_n \nabla v_n + v_n \nabla u_n|^2 = \int_U |\nabla u_n|^2 + \int_V |\nabla v_n|^2 + o(1) = 4\pi + o(1) \text{ as } n \to \infty\ ; \tag{6.8}$$

for the last equality, we use again Lemma 6.1 b) and Lemma 6.2 b). By combining (6.6) and (6.8), we find that

$$\lim_{n\to\infty} E_\kappa(w_n) = 2\pi, \tag{6.9}$$

and (6.4) follows.

Corollary 6.1. *Assume that $m_{\kappa'}$ is attained for some $\kappa' > 0$. Then m_κ is attained for $0 \leq \kappa < \kappa'$.*

Proof : Let $u_{\kappa'}$ be a minimizer of (1.1)-(1.3) for $\kappa = \kappa'$. Since $u_{\kappa'}$ satisfies

$$-\Delta u_{\kappa'} = {\kappa'}^2 u_{\kappa'} (1 - |u_{\kappa'}|^2), \tag{6.10}$$

we noticed, during the proof of Corollary 5.3, that we can not have $|u_{\kappa'}| = 1$. Therefore,

$$\int_A (1 - |u_{\kappa'}|^2)^2 > 0, \tag{6.11}$$

and thus

$$E_\kappa(u_{\kappa'}) < E_{\kappa'}(u_{\kappa'}) \le 2\pi \quad \text{ if } 0 \le \kappa < \kappa'. \tag{6.12}$$

The conclusion follows now from Proposition 5.1.

Corollary 6.2. *Assume that $m_{\kappa'}$ is not attained for some $\kappa' \ge 0$. Then $m_{\kappa'} = 2\pi$ and m_κ is not attained for $\kappa > \kappa'$.*

Corollary 6.3. *For each A, there is some $\kappa' \in [0, \infty]$ such that :*
a) m_κ is always attained for $0 \le \kappa < \kappa'$;
b) m_κ is never attained for $\kappa > \kappa'$.
If, in addition, $\kappa' < \infty$, then $m_{\kappa'} = 2\pi$ and $m_\kappa = 2\pi$ for $\kappa > \kappa'$.

Remark 6.2. *When A has a single hole, it follows from Proposition 5.2. that $\kappa' > 0$. We do not know whether, when A has more than one hole, we always have $\kappa' > 0$.*

Remark 6.3. *The conjecture mentioned in the introduction is equivalent to: $I_0 > 2\pi \Longrightarrow \kappa' < \infty$.*

6.2 Asymptotic behavior of m_κ

Among other facts, we prove below that the upper bounds established in Proposition 6.1 are asymptotically optimal as $\kappa \to \infty$.

Corollary 6.4. *Assume that $I_0 < 2\pi$. Then :*
a) $\lim\limits_{\kappa\to\infty} m_\kappa = I_0$;
b) up to subsequences, minimizers $u_\kappa \to u$ strongly in $H^1(A)$ as $\kappa \to \infty$, where u is some minimizer of (1.14)-(1.15).

Proof : First note that m_κ is non-decreasing with κ, so that the limit in a) exists. Let u be such that, along some subsequence, $u_{\kappa_n} \rightharpoonup u$ weakly in $H^1(A)$ and a.e. Since

$$\frac{1}{2}\int\limits_A |\nabla u|^2 \le \liminf_{n\to\infty} \frac{1}{2}\int\limits_A |\nabla u_{\kappa_n}|^2 \le \liminf_{n\to\infty} E_{\kappa_n}(u_{\kappa_n}) = \lim_{\kappa\to\infty} E_\kappa(u_\kappa) \le I_0 < 2\pi, \tag{6.13}$$

we find by Corollary 5.1 that $u \in \mathcal{J}$ and that

$$\frac{1}{2}\int\limits_A |\nabla u|^2 \le I_0. \tag{6.14}$$

On the other hand,

$$\int\limits_A (1 - |u_\kappa|^2)^2 \le \frac{4}{\kappa^2} E_\kappa(u_\kappa) \to 0 \quad \text{ as } \kappa \to \infty, \tag{6.15}$$

so that $|u| = 1$ a.e. Therefore, $u \in \mathcal{K}$, so that u has to be a minimizer of (1.14)-(1.15), by (6.14). Recalling (6.13) , we find that

$$\lim_{n\to\infty} \int_A |\nabla u_{\kappa_n}|^2 = \int_A |\nabla u|^2, \tag{6.16}$$

and thus $u_{\kappa_n} \to u$ strongly in $H^1(A)$. Part a) follows from (6.13) and (6.16).

Corollary 6.5. *Assume that* $I_0 = 2\pi$. *Then* $\lim_{\kappa\to\infty} m_\kappa = I_0 = 2\pi$.

Proof : Let u be such that, up to some subsequence, $u_{\kappa_n} \rightharpoonup u$ weakly in $H^1(A)$ and a.e. If $u \in \mathcal{J}$, then u is a minimizer of (1.14)-(1.15) and the conclusion follows as in the previous Corollary. If $u \notin J$, by Corollary 5.1 we find that

$$2\pi \geq \lim_{\kappa\to\infty} E_\kappa(u_\kappa) = \liminf_{n\to\infty} E_{\kappa_n}(u_{\kappa_n}) \geq \liminf_{n\to\infty} \frac{1}{2} \int_A |\nabla u_{\kappa_n}|^2 \geq 2\pi, \tag{6.17}$$

and the conclusion follows again.

Corollary 6.6. *Assume that* $I_0 > 2\pi$. *Then* $\lim_{\kappa\to\infty} m_\kappa = 2\pi$.

Proof : Due to the corrolaty 6.3 if $\kappa' < \infty$ then conlsuion follows. Thus we assume that $\kappa' = \infty$ so m_k is attained and we can consider a sequence of minimizers u_{κ_n}" Let u be such that, up to some subsequence, $u_{\kappa_n} \rightharpoonup u$ weakly in $H^1(A)$ and a.e. Since

$$\frac{1}{2}\int_A |\nabla u|^2 \leq \liminf_{n\to\infty} \frac{1}{2}\int_A |\nabla u_{\kappa_n}|^2 \leq \liminf_{n\to\infty} E_{\kappa_n}(u_{\kappa_n}) = \lim_{\kappa\to\infty} E_\kappa(u_\kappa) \leq 2\pi, \tag{6.18}$$

we cannot have $u \in \mathcal{J}$. For otherwise, as in the proof of Corollary 6.4, we would have $u \in \mathcal{K}$ and

$$2\pi < I_0 \leq \frac{1}{2}\int_A |\nabla u|^2 \leq 2\pi, \tag{6.19}$$

which is impossible. Thus $u \notin \mathcal{J}$ and the conclusion follows by combining (6.18) with Corollary 5.1.

We will need later the following refinement of (6.5)

Lemma 6.3. *There are some constants* $C = C(A) > 0$ *and* $\kappa' = \kappa'(A) > 0$ *such that*

$$m_\kappa \leq I_0 - \frac{C}{\kappa^2}, \quad \forall\ \kappa > \kappa'. \tag{6.20}$$

Proof : Let u be a minimizer of (1.14)-(1.15) and $g = \mathrm{tr}_{\partial A} u$. Let, for $\kappa > 0$, $v_\kappa \in \mathcal{J}$ be the minimizer of E_κ in the class $\mathcal{L}$ given by (5.20). We claim that

$$v_\kappa \to u \text{ strongly in } H^1(A) \text{ as } \kappa \to \infty. \tag{6.21}$$

Indeed, let v be any possible weak limit of some subsequence of (u_κ). Since $E_\kappa(v_\kappa) \le E_\kappa(u) = I_0$, we find as in the proof of Corollary 6.4 that $|v| = 1$ and $\frac{1}{2}\int\limits_A |\nabla v|^2 \le I_0$. On the other hand, we have $\mathrm{tr}_{\partial A} u = \mathrm{tr}_{\partial A} v$, so that $v \in \mathcal{K}$. Therefore, v is a minimizer of (1.14)-(1.15). By Lemma 2.4, there is some $\alpha \in S^1$ such that $v = \alpha u$. This α has to be 1, since u and v agree on ∂A. Finally, we proceed as in the proof of Corollary 6.4 to obtain the strong H^1 convergence claimed in (6.21).

We now invoke the following result in [9]

Lemma 6.4. ([9]) *We have*

$$\kappa^2(1 - |v_\kappa|^2) \to |\nabla u|^2 \quad \text{in } C_{\mathrm{loc}}(A) \text{ as } \kappa \to \infty \tag{6.22}$$

and

$$\lim_{\kappa \to \infty} E_\kappa(v_\kappa) = \frac{1}{2}\int\limits_A |\nabla u|^2 = I_0. \tag{6.23}$$

Actually, the above result was obtained in [9] for minimizers of the Ginzburg-Landau energy E_κ with a fixed Dirichlet boundary data g under the additional assumption that A is simply connected. However, the simple connectedness of A is used in their proof only for establishing (6.21). Since we proved above (6.21), we may apply Lemma 6.4 to our case.
Note that u is smooth and non constant. Therefore, we may find a compact $K \subset A$ and constants $c = c(A) > 0$, $\kappa' = \kappa'(A) > 0$ such that

$$\kappa^2(1 - |v_\kappa|^2) \ge c \quad \text{in } K \text{ for } \kappa > \kappa'(A). \tag{6.24}$$

Let now $\kappa > \kappa'(A)$. Using (6.23), we find that

$$I_0 - E_\kappa(v_\kappa) = \sum_{n \ge 0} (E_{2^{n+1}\kappa}(v_{2^{n+1}\kappa}) - E_{2^n\kappa}(v_{2^n\kappa})). \tag{6.25}$$

By the minimality of v_κ with respect to E_κ, we have

$$E_{2^{n+1}\kappa}(v_{2^{n+1}\kappa}) - E_{2^n\kappa}(v_{2^n\kappa}) \ge E_{2^{n+1}\kappa}(v_{2^{n+1}\kappa}) - E_{2^n\kappa}(v_{2^{n+1}\kappa}) = \frac{3}{4}4^n\kappa^2 \int\limits_A (1 - |v_{2^{n+1}\kappa}|^2)^2. \tag{6.26}$$

Combining (6.24) and (6.26), we find

$$E_{2^{n+1}\kappa}(v_{2^{n+1}\kappa}) - E_{2^n\kappa}(v_{2^n\kappa}) \ge \frac{3}{4}4^n\kappa^2 \int\limits_K (1 - |v_{2^{n+1}\kappa}|^2)^2 \ge 3c^2 4^{-n-3}\kappa^{-2}|K|. \tag{6.27}$$

Going back to (6.25), we obtain, for $\kappa > \kappa'(A)$, that

$$m_\kappa \leq E_\kappa(v_\kappa) \leq I_0 - \sum_{n\geq 0} 3c^2 4^{-n-3}\kappa^{-2}|K| = I_0 - \frac{C}{\kappa^2}. \tag{6.28}$$

7 Properties of quasi-minimizers

We start by defining the quasi-minimizers we alluded to in the Introduction.

Definition 7.1. *A* **family of quasi-minimizers** *is a family $(u_\kappa) \subset \mathcal{J}$, $\kappa \geq 0$, such that :*

$$E_\kappa(u_\kappa) \leq m_\kappa + \frac{1}{e^\kappa} \tag{7.1}$$

and u_κ is a minimizer of the Ginzburg-Landau energy with respect to its own boundary condition, i.e.,

$$E_\kappa(u_\kappa) \leq E_\kappa(v) \quad \text{if } \mathrm{tr}_{\partial A} v = \mathrm{tr}_{\partial A} u_\kappa. \tag{7.2}$$

Note that quasi-minimizers always exist. Moreover, any minimizer u_κ of m_κ, if it exists, is a quasi-minimizer. It also follows from the maximum principle that

$$|u_\kappa| \leq 1 \quad \text{ in } A. \tag{7.3}$$

On the other hand, quasi-minimizers satisfy

$$-\Delta u_\kappa = \kappa^2 u_\kappa (1 - |u_\kappa|^2) \quad \text{ in } A \tag{7.4}$$

and thus they are smooth in A.
Note also that, in view of Proposition 6.1, we have the following uniform bound

$$E_\kappa(u_\kappa) \leq 2\pi + 1 = C. \tag{7.5}$$

The main tool for determining the asymptotic behavior of the quasi-minimizers is the following

Lemma 7.1. ([33]) *Set, for $z \in A$, $d(z) = \mathrm{dist}(z, \partial A)$. Under the assumptions (7.3)-(7.5), we have*

$$|D^l u_\kappa(z)| \leq \frac{C_l}{d^l(z)}, \quad l \in I\!N,\ z \in A \tag{7.6}$$

and

$$|D^l(1 - |u_\kappa|^2)(z)| \leq \frac{C_l}{\kappa^2 d^{l+2}(z)}, \quad l \in I\!N,\ z \in A. \tag{7.7}$$

Here, C_l are explicit constants depending only on l and the constant C in (7.5).

Actually, the above estimates were established in [33] when $d(z) = 1$; the general case follows by scaling. Note that these estimates deteriorate when we approach the boundary ; in fact, it is not reasonable to expect uniform estimates up to the boundary.

Corollary 7.1. *Let (u_κ) be a family of quasi-minimizers. Then, up to subsequences, (u_κ) converges, weakly in $H^1(A)$ and strongly in $C^l_{\text{loc}}(A)$, $l \in I\!N$, to some $u \in C^\infty(A)$ such that $|u| = 1$.*

Proposition 7.1. *Assume that $I_0 > 2\pi$ and let (u_κ) be a family of quasi-minimizers. Then there are constants $\alpha_\kappa \in S^1$ such that $(\alpha_\kappa u_\kappa)$ converges, weakly in $H^1(A)$ and strongly in $C^l_{\text{loc}}(A)$, $l \in I\!N$, to 1.*

Proof : We start by proving that any possible limit as in Corollary 7.1 is a constant of modulus 1. Indeed, let u be any S^1-valued smooth map such that, along some subsequence, (u_κ) converges, weakly in $H^1(A)$ and strongly in $C^l_{\text{loc}}(A)$, $l \in I\!N$, to u. First of all, the proof of Corollary 6.6 implies that $u \notin \mathcal{K}$. Thus u is in some class $K_{D,\, d_0, \dots, d_k}$ with $(D, d_0, \dots, d_k) \neq (1, -1, 0, \dots, 0)$. Since $u \in H^1(A\,; S^1)$, Lemma 2.2 implies that $D + \sum_{j=0}^{k} d_j = 0$. Therefore, among the degrees $D, d_0, \dots, d_k$, there are **at least two** degrees different from the corresponding degrees $1, -1, 0, \dots, 0$. Let us assume, for simplicity, that $D \neq 1$ and $d_0 \neq -1$, the analysis being similar in the other cases. Fix some small $\delta > 0$ and let

$$\Gamma_\delta = \{z \in A\,; \text{dist}(z, \partial\Omega) = \delta\},\ \Omega_\delta = \{z \in A\,; \text{dist}(z, \partial\Omega) < \delta\}$$

and respectively

$$\gamma_\delta = \{z \in A\,; \text{dist}(z, \partial\omega_0) = \delta\},\ \omega_\delta = \{z \in A\,; \text{dist}(z, \partial\omega_0) < \delta\}.$$

We orient Γ_δ and γ_δ as boundaries of Ω_δ and ω_δ, respectively. By Lemma 2.2 applied to u in Ω_δ and in ω_δ, we have

$$\deg(u, \Gamma_\delta) = -D \quad \text{ and } \deg(u, \gamma_\delta) = -d_0. \tag{7.8}$$

We now argue as in the proof of Proposition 5.1. Assume that $u_{\kappa_l} \rightharpoonup u$ as in Corollary 7.1. Then we have

$$2\pi \geq \liminf m_{\kappa_l} \geq \liminf \frac{1}{2}\int_A |\nabla u_{\kappa_l}|^2 = \liminf \frac{1}{2}\int_A |\nabla(u_{\kappa_l} - u)|^2 + \frac{1}{2}\int_A |\nabla u|^2, \tag{7.9}$$

so that

$$2\pi \geq \liminf \left(\frac{1}{2}\int_{\Omega_\delta} |\nabla(u_{\kappa_l} - u)|^2 + \frac{1}{2}\int_{\omega_\delta} |\nabla(u_{\kappa_l} - u)|^2\right) + \frac{1}{2}\int_A |\nabla u|^2. \tag{7.10}$$

Now

$$\liminf \frac{1}{2}\int_{\Omega_\delta} |\nabla(u_{\kappa_l} - u)|^2 \geq \liminf \left|\int_{\Omega_\delta} \text{Jac }(u_{\kappa_l} - u)\right| = \liminf \frac{1}{2}\left|\int_{\partial\Omega_\delta} (u_{\kappa_l} - u) \wedge \frac{\partial(u_{\kappa_l} - u)}{\partial\tau}\right|. \tag{7.11}$$

Using the $C^1_{\mathrm{loc}}(A)$ convergence of u_{κ_l} to u, we find that

$$\liminf \frac{1}{2}\int\limits_{\Omega_\delta} |\nabla(u_{\kappa_l}-u)|^2 \geq \liminf \frac{1}{2}\Big|\int\limits_{\partial\Omega} (u_{\kappa_l}-u)\wedge \frac{\partial(u_{\kappa_l}-u)}{\partial\tau}\Big|. \tag{7.12}$$

As in the proof of Proposition 5.1, we obtain

$$\liminf \frac{1}{2}\int\limits_{\Omega_\delta} |\nabla(u_{\kappa_l}-u)|^2 \geq \pi|D-1|. \tag{7.13}$$

Similarly,

$$\liminf \frac{1}{2}\int\limits_{\omega_\delta} |\nabla(u_{\kappa_l}-u)|^2 \geq \pi|d_0+1|. \tag{7.14}$$

By combining (7.10), (7.13), (7.14) and the fact that $D\neq 1$, $d_0\neq -1$, we are led to $\int\limits_A |\nabla u|^2 = 0$, so that u is a constant of modulus 1.

We next prove the existence of the family (α_κ). Set

$$\beta_\kappa = \frac{1}{|A|}\int\limits_A u_\kappa. \tag{7.15}$$

Since any possible limit u is a constant of modulus 1, it is easy to see that $|\beta_\kappa|\to 1$ as $\kappa\to\infty$. If we set, for sufficiently large κ, $\alpha_\kappa = \dfrac{\overline{\beta_\kappa}}{|\beta_\kappa|}$, then for this choice the conclusion of the proposition is straightforward.

As a consequence of the above proposition, we obtain that quasi-minimizers have to vanish at least twice in the supercritical case

Lemma 7.2. *Assume that A is supercritcal, i.e., that $I_0 > 2\pi$. Let (u_κ) be a family of quasi-minimizers. Fix some small $\delta > 0$. Then there is some $\kappa' = \kappa'(A)$ such that, for $\kappa > \kappa'$, u_κ has at least a zero at distance $< \delta$ from $\partial\Omega$ and at least a zero at distance $< \delta$ from $\partial\omega_0$.*

Remark 7.1. We will see in Section 11 that u_κ has exactly two zeroes for large κ, but the argument is rather involved.

Proof of Lemma 7.2 : We reason near $\partial\Omega$, the situation being similar near $\partial\omega_0$. Fix some $\delta > 0$ and let $\Gamma = \{z\in A\,;\mathrm{dist}(z,\partial\Omega)=\delta\}$. Let κ' be such that, for $\kappa>\kappa'$, we have $|\alpha_\kappa u_\kappa - 1| < 1/2$ on Γ. Then, for any such κ, we have $\deg(\alpha_\kappa u_\kappa,\Gamma) = 0$, and thus $\deg(u_\kappa,\Gamma)=0$. Argue by contradiction and asume that, for such a κ, u_κ does not vanish in the domain U enclosed by Γ

and $\partial\Omega$. We claim that, in this case, there is a constant $C = C(\kappa)$ such that $C \le |u_\kappa| \le 1$ in U. Assuming the claim proved, for the moment, it follows that the map $v = u_\kappa/|u_\kappa|$ is in $H^1(U; S^1)$, and it has degrees 1 on $\partial\Omega$ and 0 on Γ. (The fact that $\nabla v \in L^2(U)$ follows from the pointwise inequality $|\nabla v| \le |\nabla u_\kappa|/C$ in U.) But this is impossible, by Lemma 2.2.
Returning to the claim, we start by noting that there is some $\varepsilon = \varepsilon_\kappa > 0$ such that $|u_\kappa(z)| \ge 1/2$ if $\text{dist}(z, \partial\Omega) > \varepsilon$ i; this follows from the proof of Lemma 4.4. On the other hand, there is some $D = D(\kappa) > 0$ such that $|u_\kappa(z)| \ge D$ when $\varepsilon \le\text{dist}(z, \partial\Omega) \le \delta$, since u_κ is smooth and non vanishing. The claim follows with $C = \text{Min}\{1/2, D\}$. The proof of the lemma is complete.

8 Asymptotic behavior of the minimizers in the subcritical case $I_0 < 2\pi$

Throughout this section, we always assume that we are in the subcritical case, i.e., that

$$I_0 < 2\pi. \tag{8.1}$$

If we combine Corollary 5.2, Corollary 6.4 and Corollary 7.1, we already know that in the subcritical case we have

$$m_\kappa < I_0 < 2\pi, \tag{8.2}$$

$$\lim_{\kappa\to\infty} m_\kappa = I_0, \tag{8.3}$$

$$m_\kappa \text{ is attained for each } \kappa \ge 0 \tag{8.4}$$

and, if u_κ is a minimizer of (1.1)-(1.3) then, up to some subsequences,

$$u_\kappa \to u \text{ strongly in } H^1(A) \text{ and in } C^l_{\text{loc}}(A), \forall\, l \in I\!N, \text{ where } u \text{ is a minimizer of } (1.14) - (1.15). \tag{8.5}$$

The aim of this section is to improve the property (8.5), by proving that u_κ converges to u in some better spaces.

We start with a preliminary remark : the minimization problem (1.1)-(1.3 is degenerate, in the sense that, if u_κ is a minimizer, so is αu_κ, for $\alpha \in S^1$. It will be convenient to reduce degeneracy by replacing each u_κ with some appropriate rotation of u_κ. This is done in

Lemma 8.1. *Assume $I_0 < 2\pi$ and let, for each $\kappa \ge 0$, u_κ be a minimizer of (1.1)-(1.3). Fix some minimizer u_∞ of I_0. Then there is a family $(\alpha_\kappa) \subset S^1$ such that*

$$v_\kappa = \alpha_\kappa u_\kappa \to u_\infty \textit{ strongly in } H^1(A) \textit{ and in } C^l_{\text{loc}}(A), \forall\, l \in I\!N. \tag{8.6}$$

Proof : We proceed as at the end of the proof of Proposition 7.1. Set

$$\beta_\kappa = \frac{1}{|A|}\int_A \overline{u_\kappa} u_\infty. \tag{8.7}$$

We first claim that $|\beta_\kappa| \to 1$ as $\kappa \to \infty$. Indeed, by (8.5), up to some subsequence we have $u_\kappa \to u$ in $H^1(A)$ for some minimizer u of I_0. By Lemma 2.4, there is some $\alpha \in S^1$ such that $u = \alpha u_\infty$, and thus $\beta_\kappa \to \overline{\alpha}$ along such a subsequence. Obviously, this implies that $|\beta_\kappa| \to 1$ as $\kappa \to \infty$. Now let

$$\alpha_\kappa = \frac{\beta_\kappa}{|\beta_\kappa|}, \tag{8.8}$$

which is well-defined for sufficiently large κ. Then, clearly,

$$\int_A \overline{\alpha_\kappa u_\kappa} u_\infty \to 1 \quad \text{ as } \kappa \to \infty. \tag{8.9}$$

Invoking once again (8.5) and Lemma 2.4, we find that (8.6) holds for this choice of α_κ.

The key ingredient in improving (8.5) is the following

Lemma 8.2. *We have*

$$|u_\kappa| \to 1 \text{ uniformly in } \overline{A} \text{ as } \kappa \to \infty. \tag{8.10}$$

Proof : It suffices to work with v_κ instead of u_κ. Recall that $|v_\kappa| \leq 1$, by (7.3), so that, for $0 < a < 1$, it suffices to prove that there is some $\kappa^a \geq 0$ such that

$$|v_\kappa| \geq a \text{ in } \overline{A}, \quad \forall\ \kappa \geq \kappa^a. \tag{8.11}$$

We start by noting that (8.11) holds "far away" from ∂A. Indeed, by (7.7) in Lemma 7.1, we have

$$|v_\kappa(z)| \geq a \quad \text{ if } d(z) \geq \sqrt{\frac{C_0}{\kappa^2(1-a^2)}} = \frac{C^a}{\kappa}, \tag{8.12}$$

where $d(z) = \text{dist}(z, \partial A)$.
We next prove that (8.11) holds "near" ∂A. For this purpose, we need the following

Lemma 8.3. ([20]) *Let* $(g^n) \subset \mathrm{VMO}(\partial A; S^1)$ *be such that* $g^n \to g$ *strongly in* $\mathrm{VMO}(\partial A)$. *Then, for each* $0 < a < 1$, *there is some* $\delta > 0$ **independent of** n *such that*

$$a \leq |\widetilde{g^n}(z)| \leq 1 \quad \text{ if } d(z) < \delta. \tag{8.13}$$

Here, $\widetilde{g^n}$ *is the harmonic extension of* g^n *to* A.

Proof of Lemma 8.2 continued : Set $g_\kappa = \mathrm{tr}_{\partial A} v_\kappa$ and $g = \mathrm{tr}_{\partial A} u_\infty$. Since $v_\kappa \to u_\infty$ strongly in $H^1(A)$, we have $g_\kappa \to g$ strongly in $H^{1/2}(\partial A)$. Using the fact that $H^{1/2} \hookrightarrow$VMO in 1-D, we find from Lemma 8.1 and Lemma 8.3 that there are some $\delta > 0$, κ_1^a such that

$$\frac{1+a}{2} \leq |\widetilde{g_\kappa}(z)| \leq 1 \quad \text{ if } d(z) < \delta \text{ and } \kappa \geq \kappa_1^a. \tag{8.14}$$

Write now $v_\kappa = \widetilde{g_\kappa} + w_\kappa$, where w_κ satisfies

$$\begin{cases} -\Delta w_\kappa & = \kappa^2 v_\kappa(1-|v_\kappa|^2) \quad \text{in } A \\ \quad w_\kappa & = 0 \qquad\qquad\qquad\quad\ \text{on } \partial A \end{cases} . \tag{8.15}$$

In order to estimate $|w_\kappa|$, we rely on the following

Lemma 8.4. ([9]) *Let $u \in C^2(\overline{A})$ satisfy*

$$\begin{cases} \Delta u & = f \quad \text{in } A \\ \ u & = 0 \quad \text{on } \partial A \end{cases} . \tag{8.16}$$

Then, for some constant C_A depending only on A, we have

$$\|\nabla u\|_{L^\infty(A)} \le C_A \|\ u\|^{1/2}_{L^\infty(A)} \|\ f\|^{1/2}_{L^\infty(A)}. \tag{8.17}$$

Proof of Lemma 8.2 completed : Since $|v_\kappa| \le 1$, by (7.3), and $|\widetilde{g_\kappa}(z)| \le 1$ (as harmonic extension of a map of modulus 1), we find that

$$|w_\kappa| \le 2 \quad \text{in } A \tag{8.18}$$

and

$$|\Delta w_\kappa| \le \kappa^2 \quad \text{in } A. \tag{8.19}$$

By combining (8.18), (8.19) and Lemma 8.4, we obtain

$$|\nabla w_\kappa| \le \sqrt{2} C_A \kappa \quad \text{in } A. \tag{8.20}$$

Recalling that $w_\kappa = 0$ on ∂A, we obtain, for some constant D_A depending only on A, that

$$|w_\kappa(z)| \le D_A \kappa\, d(z) \quad \text{in } A. \tag{8.21}$$

By combining (8.14) and (8.21) we find, for large κ,

$$|v_\kappa(z)| \ge a \quad \text{if } d(z) \le \frac{1-a}{2D_a\kappa} = \frac{D^a}{\kappa}. \tag{8.22}$$

We complete the proof of Lemma 8.2 by establishing that (8.11) holds in the region of A uncovered by the estimates (8.12) and (8.22). This part of the proof follows [9]. We may assume that $D^a \le C^a$, for otherwise the whole of A is covered by these estimates. First, note that, by applying (7.6) to v_κ, we have

$$|\nabla v_\kappa(z)| \le E_A \kappa \quad \text{if } \frac{D^a}{2\kappa} \le d(z) \le \frac{C^a}{\kappa}, \tag{8.23}$$

where E_A is independent of large κ. Assume that there is some $z \in A$ such that

$$|v_\kappa(z)| \le a \quad \text{and} \quad \frac{D^a}{\kappa} \le d(z) \le \frac{C^a}{\kappa}. \tag{8.24}$$

¿From (8.23) and (8.24), we derive the existence of some $c^a < D^a$ such that

$$|v_\kappa(y)| \le \frac{1+a}{2} \quad \text{if} \quad |y-z| \le \frac{c^a}{\kappa}. \tag{8.25}$$

Pick now κ_2^a such that

$$\{y\,;|y-z| \le \frac{c^a}{\kappa}\} \subset A \quad \text{if } \frac{D^a}{\kappa} \le d(z) \le \frac{C^a}{\kappa} \text{ and } \kappa \ge \kappa_2^a. \tag{8.26}$$

Then

$$\frac{\kappa^2}{4}\int_A (1-|v_\kappa|^2)^2 \ge \frac{\kappa^2}{4}\int_{\{y;|y-z|\le c^a/\kappa\}} (1-|v_\kappa|^2)^2 \ge \frac{\pi(c^a)^2(1-a)(1+3a)}{4} \quad \text{if } \kappa \ge \kappa_2^a. \tag{8.27}$$

We will finally prove that (8.27) (and thus (8.24)) can not hold for sufficiently large κ. (This will complete the proof, in view of (8.12) and (8.22).) To this purpose, it suffices to establish that

$$\lim_{\kappa\to\infty} \frac{\kappa^2}{4}\int_A (1-|v_\kappa|^2)^2 = 0. \tag{8.28}$$

This is an easy consequence of (8.3) and (8.5), since

$$\frac{\kappa^2}{4}\int_A (1-|v_\kappa|^2)^2 = m_\kappa - \frac{1}{2}\int_A |\nabla v_\kappa|^2 \to I_0 - \frac{1}{2}\int_A |\nabla u_\infty|^2 = 0 \quad \text{as } \kappa\to\infty. \tag{8.29}$$

One of the useful consequences of Lemma 8.2 is that it allows us to rewrite, for large values of κ, the Ginzburg-Landau equation in terms of the modulus and the phase of u_κ. Let κ_2 be such that $|u_\kappa| \ge 1/2$ for $\kappa \ge \kappa_2$. Set $\rho = \rho_\kappa = |u_\kappa|$, for $\kappa \ge \kappa_2$. Then $\dfrac{u_\kappa}{\rho} \in \mathcal{J}$. By Lemma 2.3 a), we may write $\dfrac{u_\kappa}{\rho} = u_\infty e^{\imath\psi}$ for some $\psi = \psi_\kappa \in H^1(A;\mathbb{R})$; actually, $\rho,\psi \in C^\infty(\overline{A};\mathbb{R})$, by Lemma 4.4. Moreover, we noted in the statement of Lemma 2.4 that $u_\infty = e^{\imath\varphi}$, where φ is only locally defined, but it has a globally defined gradient. We may thus write, locally in A,

$$u_\kappa = \rho e^{\imath(\varphi+\psi)} = \rho_\kappa e^{\imath(\varphi+\psi_\kappa)} = \rho_\kappa e^{\imath\psi_\kappa} u_\infty, \tag{8.30}$$

the last expression in (8.30) being globally defined. It follows immediately from (4.16) and (4.17) that ψ and ρ satisfy

$$\begin{cases} -\Delta\rho &= \kappa^2\rho(1-\rho^2) - \rho|\nabla(\varphi+\psi)|^2 \quad \text{in } A \\ \quad\ \rho &= 1 \quad \text{on } \partial A \end{cases} \tag{8.31}$$

and respectively

$$\left\{ \begin{array}{lll} -\mathrm{div}(\rho^2\nabla\psi) & = \mathrm{div}(\rho^2\nabla\varphi) = 2\rho\nabla\rho\cdot\nabla\varphi & \text{in } A \\ \dfrac{\partial\psi}{\partial\nu} & = 0 & \text{on } \partial A \end{array} \right. . \tag{8.32}$$

Two remarks about the equation of ψ : on the one hand, it has a global meaning, since $\nabla\varphi$ is globally defined. On the other hand, the second equality in (8.32) comes from the fact, by Lemma 2.4, we have $\mathrm{div}(\nabla\varphi) = \mathrm{div}(-\partial\eta/\partial y, \partial\eta/\partial x) = 0$.

Lemma 8.5. *We have*

$$\psi_\kappa - \frac{1}{|A|}\int_A \psi_\kappa \to 0 \quad \textit{in } W^{1,p}(A),\ 1 < p < \infty,\ \textit{as } \kappa \to \infty. \tag{8.33}$$

Proof : We rewrite (8.32) as

$$\left\{ \begin{array}{lll} \Delta\psi & = \mathrm{div}((1-\rho^2)\nabla(\varphi+\psi)) & \text{in } A \\ \dfrac{\partial\psi}{\partial\nu} & = 0 & \text{on } \partial A \end{array} \right. . \tag{8.34}$$

By standard elliptic estimates ([26]) we have, for $1 < p < \infty$,

$$\|\nabla\psi\|_{L^p(A)} \le C_p\|(1-\rho^2)\nabla(\varphi+\psi)\|_{L^p(A)} \le C_p\|1-\rho^2\|_{L^\infty(A)}(\|\nabla\varphi\|_{L^p(A)} + \|\nabla\psi\|_{L^p(A)}). \tag{8.35}$$

By Lemma 8.2, we have $C_p\|1-\rho^2\|_{L^\infty(A)} \le 1/2$ for sufficiently large κ, and thus, for such κ, we obtain

$$\|\nabla\psi\|_{L^p(A)} \le 2C_p\|1-\rho^2\|_{L^\infty(A)}\|\nabla\varphi\|_{L^p(A)} \to 0 \quad \text{as } \kappa \to \infty. \tag{8.36}$$

The conclusion of Lemma 8.5 follows from (8.36) and the Poincaré-Wirtinger inequality.

For the next result, we follow the unpublished paper [34].

Lemma 8.6. *We have, for sufficiently large κ and a constant C_A depending only on A, the estimate*

$$\int_A (1-\rho_\kappa^2)^2 \le \frac{C_A}{\kappa^4}. \tag{8.37}$$

Proof : A straightforward computation shows that

$$E_\kappa(u_\kappa) = \frac{1}{2}\int_A |\nabla\rho|^2 + \frac{1}{2}\int_A \rho^2|\nabla\varphi|^2 + \frac{1}{2}\int_A \rho^2|\nabla\psi|^2 + \int_A \rho^2\nabla\varphi\cdot\nabla\psi + \frac{\kappa^2}{4}\int_A (1-\rho^2)^2. \tag{8.38}$$

¿From (2.21), it follows that $\nabla\varphi$ satisfies

$$\left\{ \begin{array}{lll} \mathrm{div}(\nabla\varphi) & = 0 & \text{in } A \\ \nu\cdot\nabla\varphi & = 0 & \text{on } \partial A \end{array} \right. . \tag{8.39}$$

If we multiply (8.39) by ψ and integrate, we find that

$$\int_A \nabla\varphi \cdot \nabla\psi = 0. \tag{8.40}$$

By combining (8.38) and (8.40), we are led to

$$E_\kappa(u_\kappa) = \frac{1}{2}\int_A |\nabla\rho|^2 + \frac{1}{2}\int_A \rho^2|\nabla\varphi|^2 + \frac{1}{2}\int_A \rho^2|\nabla\psi|^2 + \int_A (\rho^2-1)\nabla\varphi\cdot\nabla\psi + \frac{\kappa^2}{4}\int_A (1-\rho^2)^2. \tag{8.41}$$

We now use the minimality of u_κ. Since $u_\infty \in \mathcal{J}$, we have

$$m_\kappa = E_\kappa(u_\kappa) \le E_\kappa(u_\infty) = I_0 = \frac{1}{2}\int_A |\nabla\varphi|^2. \tag{8.42}$$

Using (8.41) together with (8.42), we obtain

$$\frac{1}{2}\int_A |\nabla\rho|^2 + \frac{1}{2}\int_A \rho^2|\nabla\psi|^2 + \frac{\kappa^2}{4}\int_A (1-\rho^2)^2 \le \frac{1}{2}\int_A (1-\rho^2)|\nabla\varphi|^2 + \int_A (1-\rho^2)\nabla\varphi\cdot\nabla\psi. \tag{8.43}$$

We estimate the integrals on the right-hand side of (8.43) using the Cauchy-Schwartz inequality and obtain

$$\frac{1}{2}\int_A (1-\rho^2)|\nabla\varphi|^2 \le \frac{\kappa^2}{16}\int_A (1-\rho^2)^2 + \frac{4}{\kappa^2}\int_A |\nabla\varphi|^4 \tag{8.44}$$

and respectively

$$\int_A (1-\rho^2)\nabla\varphi\cdot\nabla\psi \le \frac{\kappa^2}{16}\int_A (1-\rho^2)^2 + \frac{4}{\kappa^2}\int_A |\nabla\psi|^2|\nabla\varphi|^2. \tag{8.45}$$

Combining (8.43), (8.44) and (8.45), we find

$$\frac{1}{2}\int_A |\nabla\rho|^2 + \frac{1}{2}\int_A (\rho^2 - \frac{4}{\kappa^2}|\nabla\varphi|^2)|\nabla\psi|^2 + \frac{\kappa^2}{8}\int_A (1-\rho^2)^2 \le \frac{4}{\kappa^2}\int_A |\nabla\varphi|^4. \tag{8.46}$$

Recall that, by Lemma 2.4, $\nabla\varphi$ is smooth in $\overline{A}$, and thus bounded. Recall also that $\rho \to 1$ uniformly in $\overline{A}$, by Lemma 8.2. Therefore, for sufficiently large κ, we have the estimate

$$\frac{1}{2}\int_A |\nabla\rho|^2 + \frac{1}{3}\int_A |\nabla\psi|^2 + \frac{\kappa^2}{8}\int_A (1-\rho^2)^2 \le \frac{C}{\kappa^2}, \tag{8.47}$$

and the conclusion of Lemma 8.6 follows.

Corollary 8.1. *ρ_κ remains bounded in $H^2(A)$ as $\kappa \to \infty$.*

Proof : Set $f_\kappa = \kappa^2\rho(1-\rho^2) - \rho|\nabla(\varphi+\psi)|^2$. Then (8.31) may be rewritten as

$$\begin{cases} -\Delta\rho &= f_\kappa \quad \text{in } A \\ \rho &= 1 \quad \text{on } \partial A \end{cases}. \tag{8.48}$$

Using Lemma 8.5, Lemma 8.6 and the inequality $|\rho| \le 1$ together with the fact that $|\nabla\varphi|$ is bounded in $\overline{A}$, we find that

$$\|f_\kappa\|_{L^2(A)} \le C, \qquad \text{for sufficiently large } \kappa. \tag{8.49}$$

The conclusion of the corollary follows now from (8.48) and (8.49), using standard elliptic estimates ([26]).

Estimates (8.10), (8.33) and (8.37) are the key ingredients that allow us to follow from now on the strategy developed in [9] in order to obtain further asymptotic estimates. This is done in the remaining part of this section.

Lemma 8.7. *ψ_κ remains bounded in $W^{2,p}(A)$, $1 < p < \infty$, as $\kappa \to \infty$.*

Proof : We write, for sufficiently large κ, (8.32) as

$$\begin{cases} \Delta\psi &= -\dfrac{2}{\rho}\nabla\rho\cdot\nabla(\varphi+\psi) = g_\kappa \quad \text{in } A \\ \dfrac{\partial\psi}{\partial\nu} &= 0 \quad \text{on } \partial A \end{cases}. \tag{8.50}$$

Using (8.33) and (8.37) together with the fact that $|\nabla\varphi|$ is bounded in $\overline{A}$, we obtain that g_κ remains bounded in $L^p(A)$, $1 < p < \infty$, as $\kappa \to \infty$, so that the conclusion follows.

Lemma 8.8. *ρ_κ remains bounded in $W^{2,p}(A)$, $1 < p < \infty$, as $\kappa \to \infty$.*

Proof : By Corollary 8.1, the conclusion is clear when $p \le 2$. As in the proof of Corollary 8.1, it suffices to establish, for sufficiently large κ, the estimate

$$\int_A (1-\rho^2)^p \le \frac{C}{\kappa^{2p}}, \qquad 2 < p < \infty. \tag{8.51}$$

Set $w = w_\kappa = 1 - \rho_\kappa$. Using the fact that $|u_\kappa| \le 1$, we see that $w_\kappa \ge 0$. On the other hand, we may rewrite (8.31) as

$$\begin{cases} -\Delta w + \kappa^2\rho(1+\rho)w &= \rho|\nabla(\varphi+\psi)|^2 \quad \text{in } A \\ w &= 0 \quad \text{on } \partial A \end{cases}. \tag{8.52}$$

By (8.10), for sufficiently large κ we have

$$\begin{cases} -\Delta w + \kappa^2 w & \leq |\nabla(\varphi + \psi)|^2 = h = h_\kappa \quad \text{in } A \\ \qquad\qquad w & = 0 \qquad\qquad\qquad\qquad\qquad\ \ \text{on } \partial A \end{cases} . \tag{8.53}$$

Let $p > 2$. If we multiply (8.53) by w^{p-1} and integrate, we find

$$(p-1)\int_A w^{p-2}|\nabla w|^2 + \kappa^2 \int_A w^p \leq \int_A h w^{p-1} \leq \Big(\int_A w^p\Big)^{(p-1)/p}\Big(\int_A h^p\Big)^{1/p}. \tag{8.54}$$

Hence

$$\int_A w^p \leq \frac{1}{\kappa^{2p}} \int_A h^p. \tag{8.55}$$

Using (8.33), (8.55), the inequality $\rho \leq 1$ and the fact that $|\nabla\varphi|$ is bounded in $\overline{A}$, we find

$$\int_A (1-\rho^2)^p \leq 4\int_A w^p \leq \frac{C}{\kappa^{2p}}, \tag{8.56}$$

which is the desired conclusion.

Corollary 8.2. *Let $\alpha_\kappa \in S^1$ be defined as in Lemma 8.1. Then*

$$\alpha_\kappa u_\kappa \to u_\infty \quad \text{in } C^{1,\beta}(\overline{A}),\ 0 < \beta < 1,\ \text{as } \kappa \to \infty. \tag{8.57}$$

Proof : By Lemma 8.7 and Lemma 8.8, ρ_κ and ψ_κ are bounded in $W^{2,p}(A)$, $1 < p < \infty$. For a given β such that $0 < \beta < 1$, pick some p such that $1 - 2/p > \beta$. For such a choice of p, the embedding $W^{2,p}(A) \hookrightarrow C^{1,\beta}(\overline{A})$ is compact. Since $u_\infty \in C^\infty(\overline{A})$, we find that, up to some subsequence and for some u, $v_\kappa = \rho_\kappa e^{\imath\psi_\kappa} u_\infty \to u$ in $C^{1,\beta}(\overline{A})$ as $\kappa \to \infty$. By Lemma 8.1, the limit u must coincide with u_∞, and the conclusion follows immediately.

We end by noting that all the further estimates obtained in [9] may be also established in our case by straightforward adaptations of the arguments therein.

9 Asymptotic behavior of the minimizers in the critical case $I_0 = 2\pi$

The aim of this section is to extend the results obtained in the previous section to the critical case $I_0 = 2\pi$. If we examine the proof of the estimates obtained in the subcritical case, we see that the only point where the subcriticality intervenes is via (8.5), more specifically, via the fact that the family (u_κ) converges, up to subsequences and strongly in $H^1(A)$, to some minimizer u of (1.14)-(1.15) . We will obtain below that, in the critical case, minimizers u_κ of(1.1) converge in

$H^1(A)$, up to subsequences, to some minimizer u of (1.14)-(1.15). Once this will be done, all the results in Section 8 will then automatically follow when $I_0 = 2\pi$.

We start by proving a weaker fact, namely

Lemma 9.1. *Assume $I_0 = 2\pi$. Let $u \in H^1(A; S^1) \cap C^\infty(A)$ be such that, up to some subsequence, u_κ converges to u, weakly in $H^1(A)$ and strongly in $C^l_{\mathrm{loc}}(A)$, $l \in I\!N$. Then :*
either
a) u is a minimizer of (1.14)-(1.15)
or
b) u is a constant of modulus 1.

Proof : Assume first that $u \in \mathcal{K}$. Then, by Corollary 6.5, we have, along some subsequence such that $u_{\kappa_n} \rightharpoonup u$ weakly in $H^1(A)$,

$$I_0 = 2\pi = \lim_{\kappa\to\infty} m_\kappa \geq \lim_{n\to\infty} \frac{1}{2}\int_A |\nabla u_{\kappa_n}|^2 \geq \frac{1}{2}\int_A |\nabla u|^2 \geq I_0, \tag{9.1}$$

so that u has to be a minimizer of (1.14)-(1.15).
Assume next that $u \notin \mathcal{K}$. Let $D, d_0, \ldots, d_k$ be integers such that $u \in K_{D,\, d_0, \ldots,\, d_k}$. By Lemma 2.2 and the fact that $|u| = 1$ in A, we find that $D + \sum_{j=0}^{k} d_j = 0$. Since $u \notin \mathcal{K}$, this implies that

$$|D-1| + |d_0+1| + \sum_{j=0}^{k} |d_k| \geq 2. \tag{9.2}$$

By (5.15), (9.2) and Corollary 6.5 we have

$$2\pi \geq \liminf_{n\to\infty} \frac{1}{2}\int_A |\nabla u_{\kappa_n}|^2 \geq \pi\Big(|D-1| + |d_0+1| + \sum_{j=0}^{k} |d_k|\Big) + \frac{1}{2}\int_A |\nabla u|^2 \geq 2\pi + \frac{1}{2}\int_A |\nabla u|^2, \tag{9.3}$$

so that u has to be a constant of modulus 1.

We next exclude possibility b) in Lemma 9.1.

Lemma 9.2. *Assume $I_0 = 2\pi$. Let $u \in H^1(A; S^1) \cap C^\infty(A)$ be such that, up to some subsequence, u_κ converges to u, weakly in $H^1(A)$ and strongly in $C^l_{\mathrm{loc}}(A)$, $l \in I\!N$. Then u is a minimizer of (1.14)-(1.15) in $\mathcal{K}$.*

Proof : Argue by contradiction and assume that there is some subsequence such that $u_{\kappa_n} \rightharpoonup \alpha$ weakly in $H^1(A)$ and strongly in $C^l_{\mathrm{loc}}(A)$, $l \in I\!N$ for some constant $\alpha \in S^1$. For simplicity, we drop from now on the subscript n.

Let, for $\delta > 0$ fixed and sufficiently small, $\Gamma = \{z \in A\,;\, \mathrm{dist}(z, \partial\Omega) = \delta\}$. Let U be the domain enclosed by $\partial\Omega$ and Γ and let $V = A \setminus \overline{U}$. Then

$$\partial U = \partial\Omega \cup \Gamma \quad \text{ and } \partial V = \Gamma \bigcup \cup_{j=0}^{k} \partial\omega_j, \tag{9.4}$$

provided $\delta > 0$ is sufficiently small. We orient Γ as part of ∂U.
By Lemma 7.1, Corollary 7.1 and the fact that u is a constant, we may find some κ_2 such that

$$\frac{1}{2} \le |u_\kappa(z)| \le 1 \quad \text{ and } |\nabla u_\kappa(z)| \le \frac{1}{2\pi|\Gamma|} \quad \text{ if } z \in \Gamma \text{ and } \kappa \ge \kappa_2. \tag{9.5}$$

For $\kappa \ge \kappa_2$, we may write, locally on Γ, $u_\kappa = \rho_\kappa e^{\imath\varphi_\kappa}$, where $\nabla\varphi_\kappa$ is globally defined on Γ. Moreover, we have

$$|\nabla u_\kappa|^2 = |\nabla\rho_\kappa|^2 + \rho_\kappa^2 |\nabla\varphi_\kappa|^2 \ge \frac{1}{4}|\nabla\varphi_\kappa|^2 \quad \text{ on } \Gamma. \tag{9.6}$$

Therefore,

$$|\nabla\varphi_\kappa| \le \frac{1}{\pi|\Gamma|} \quad \text{ on } \Gamma. \tag{9.7}$$

Thus

$$2\pi\,|\deg(u_\kappa, \Gamma)| = \Big|\int_\Gamma \frac{\partial\varphi_\kappa}{\partial\tau}\Big| \le \int_\Gamma |\nabla\varphi_\kappa| \le \pi. \tag{9.8}$$

Hence, for $\kappa \ge \kappa_2$, we have $\deg(u_\kappa, \Gamma) = 0$, and thus

$$\int_\Gamma \frac{\partial\varphi_\kappa}{\partial\tau} = 0 \quad \text{ for } \kappa \ge \kappa_2. \tag{9.9}$$

We next use the pointwise inequality $|\nabla u_\kappa|^2 \ge 2|\mathrm{Jac}\ u_\kappa|$, Lemma 2.1 and the degree formula (1.5) in order to obtain

$$\frac{1}{2}\int_U |\nabla u_\kappa|^2 \ge \int_U \mathrm{Jac}\ u_\kappa = \frac{1}{2}\int_{\partial\Omega} u_\kappa \wedge \frac{\partial u_\kappa}{\partial\tau} + \frac{1}{2}\int_\Gamma u_\kappa \wedge \frac{\partial u_\kappa}{\partial\tau} = \pi + \frac{1}{2}\int_\Gamma \rho_\kappa^2 \frac{\partial\varphi_\kappa}{\partial\tau} \tag{9.10}$$

and similarly

$$\frac{1}{2}\int_V |\nabla u_\kappa|^2 \ge -\int_V \mathrm{Jac}\ u_\kappa = \pi + \frac{1}{2}\int_\Gamma \rho_\kappa^2 \frac{\partial\varphi_\kappa}{\partial\tau}. \tag{9.11}$$

By combining (9.9), (9.10) and (9.11) we find

$$\frac{1}{2}\int_A |\nabla u_\kappa|^2 \ge 2\pi + \int_\Gamma \rho_\kappa^2 \frac{\partial\varphi_\kappa}{\partial\tau} = 2\pi + \int_\Gamma (\rho_\kappa^2 - 1)\frac{\partial\varphi_\kappa}{\partial\tau}, \tag{9.12}$$

so that, by (9.6), we have

$$m_\kappa \geq \frac{1}{2}\int\limits_A |\nabla u_\kappa|^2 \geq 2\pi - \int\limits_\Gamma (1-\rho_\kappa^2)|\nabla\varphi_\kappa| \geq 2\pi - 2\int\limits_\Gamma (1-\rho_\kappa^2)|\nabla u_\kappa| \quad \text{for } \kappa \geq \kappa_2. \tag{9.13}$$

Invoking (7.7), we find, with some constant C independent of κ, the estimate

$$m_\kappa \geq 2\pi - \frac{C}{\kappa^2}\int\limits_\Gamma |\nabla u_\kappa| \quad \text{for } \kappa \geq \kappa_2. \tag{9.14}$$

Recalling that we argued by contradiction and supposed that u_κ converges in $C^l_{\text{loc}}(A)$, $l \in I\!N$ to some constant, we obtain from (9.14) and the upper bound (6.20) given by Lemma 6.3 that, for some constants κ_3 and $D > 0$, we have

$$2\pi - o\Big(\frac{1}{\kappa^2}\Big) \leq m_\kappa \leq 2\pi - \frac{D}{\kappa^2} \quad \text{for } \kappa \geq \kappa_3. \tag{9.15}$$

We obtain a contradiction for sufficiently large κ. This completes the proof of Lemma 9.2.

Once we know that the only possible limits (in the sense of Corollary 7.1) of (u_κ) are the minimizers of (1.14)-(1.15), we may repeat the proof of Corollary 6.4 and obtain the following

Lemma 9.3. *Assume $I_0 = 2\pi$. Then, along subsequences, $u_\kappa \to u$ strongly in $H^1(A)$ as $\kappa \to \infty$, where u is some minimizer of (1.14)-(1.15).*

The conclusion of Lemma 9.3 suffices for obtaining, in the critical case $I_0 = 2\pi$, all the results proved in Section 8. For the convenience of the reader, we state these results as

Corollary 9.1. *Assume $I_0 = 2\pi$. Fix some minimizer u_∞ of I_0 in $\mathcal{K}$. Then :*
a) there is a family $(\alpha_\kappa) \subset S^1$ such that

$$v_\kappa = \alpha_\kappa u_\kappa \to u_\infty \text{ strongly in } H^1(A) \text{ and in } C^l_{\text{loc}}(A), \forall\, l \in I\!N\,; \tag{9.16}$$

b) u_κ remains bounded in $W^{2,p}(A)$, $1 < p < \infty$, as $\kappa \to \infty$;
c) $\alpha_\kappa u_\kappa \to u_\infty$ in $C^{1,\beta}(\overline{A})$, $0 < \beta < 1$, as $\kappa \to \infty$.

10 "Uniqueness" of the minimizers in the subcritical and critical case

10.1 Uniqueness modulo a phase shift

As already noticed in the Introduction, if u_κ is a minimizer of (1.1)-(1.3), so is αu_κ for $\alpha \in S^1$. This is why we can, at best, prove uniqueness modulo a phase shift. The following result asserts uniqueness modulo a phase shift. Its proof is based on a technique developed in [21] ; see also [30] or [34].

Proposition 10.1. *Assume that A is either subcritical or critical, i.e., that $I_0 \leq 2\pi$. Then there is some κ_0 such that, if $\kappa \geq \kappa_0$ and u_κ, v_κ are two minimizers of (1.1)-(1.3), then $v_\kappa = \alpha u_\kappa$ for some $\alpha \in S^1$.*

Proof : Let κ_2 be such that $1/2 \leq |u_\kappa|, |v_\kappa| \leq 1$ if $\kappa \geq \kappa_2$. For such κ we may write $v_\kappa = u_\kappa \rho_\kappa w_\kappa = u_\kappa \rho w$, where $w \in H^1(A; S^1)$ and $\rho = |v_\kappa|/|u_\kappa|$. It is obvious that $w \in K_{0,\,0,\,\dots,\,0}$. Applying (2.2) with $v = 1$, we find that $w_\kappa = e^{\imath\psi_\kappa} = e^{\imath\psi}$ for some globally defined $\psi \in H^1(A; I\!\!R)$. To summarize, we have

$$v_\kappa = u_\kappa \rho_\kappa e^{\imath\psi_\kappa} = u_\kappa \rho e^{\imath\psi}, \quad \text{with } \rho = \frac{|v_\kappa|}{|u_\kappa|},\ \frac{1}{2} \leq \rho \leq 2,\ \psi \in H^1(A; I\!\!R), \quad \text{ if } \kappa \geq \kappa_2. \tag{10.1}$$

On the other hand, we may write, locally in A,

$$u_\kappa = \zeta_\kappa e^{\imath\varphi_\kappa} = \zeta e^{\imath\varphi}, \quad \text{with } \zeta = |u_\kappa|,\ \frac{1}{2} \leq \zeta \leq 1,\ \nabla\varphi \text{ globally defined in } A, \quad \text{ if } \kappa \geq \kappa_2. \tag{10.2}$$

We may now invoke the following

Lemma 10.1. ([21]) *We have, for $\kappa \geq \kappa_2$,*

$$E_\kappa(v_\kappa) = E_\kappa(u_\kappa) + \frac{1}{2}\int_A \zeta^2|\nabla\rho|^2 + \frac{1}{2}\int_A \zeta^2\rho^2|\nabla\psi|^2 + \int_A \zeta^2\rho^2\nabla\varphi\cdot\nabla\psi + \frac{\kappa^2}{4}\int_A \zeta^4(1-\rho^2)^2. \tag{10.3}$$

Proof of Proposition 10.1 completed : Since $|u_\kappa| \geq 1/2$ in A for $\kappa \geq \kappa_2$, (4.17) becomes, for u_κ and $\kappa \geq \kappa_2$,

$$\begin{cases} -\mathrm{div}(\zeta^2\nabla\varphi) &= 0 \quad \text{in } A \\ \nu\cdot\nabla\varphi &= 0 \quad \text{on } \partial A \end{cases}. \tag{10.4}$$

If we multiply (10.4) by ψ and integrate, we find

$$\int_A \zeta^2\nabla\varphi\cdot\nabla\psi = 0. \tag{10.5}$$

By combining (10.3) and (10.5), we obtain

$$E_\kappa(v_\kappa) = E_\kappa(u_\kappa) + \frac{1}{2}\int_A \zeta^2|\nabla\rho|^2 + \frac{1}{2}\int_A \zeta^2\rho^2|\nabla\psi|^2 + \int_A \zeta^2(\rho^2-1)\nabla\varphi\cdot\nabla\psi + \frac{\kappa^2}{4}\int_A \zeta^4(1-\rho^2)^2. \tag{10.6}$$

By Cauchy-Schwartz, we have

$$\Big|\int_A \zeta^2(\rho^2-1)\nabla\varphi\cdot\nabla\psi\Big| \leq \frac{\kappa^2}{8}\int_A \zeta^4(1-\rho^2)^2 + \frac{2}{\kappa^2}\int_A |\nabla\varphi|^2|\nabla\psi|^2. \tag{10.7}$$

By (9.6), (10.7) and the fact that, by Corollary 8.2 and Corollary 9.1, u_κ remains bounded in $C^1(\overline{A})$, we find that

$$\Big|\int_A \zeta^2(\rho^2-1)\nabla\varphi\cdot\nabla\psi\Big| \le \frac{\kappa^2}{8}\int_A \zeta^4(1-\rho^2)^2 + \frac{C}{\kappa^2}\int_A |\nabla\psi|^2 \tag{10.8}$$

for some constant C independent of sufficiently large κ. Inserting (10.8) into (10.6), we obtain

$$E_\kappa(v_\kappa) \ge E_\kappa(u_\kappa) + \frac{1}{2}\int_A \zeta^2|\nabla\rho|^2 + \frac{1}{2}\int_A \Big(\zeta^2\rho^2 - \frac{2C}{\kappa^2}\Big)|\nabla\psi|^2 + \frac{\kappa^2}{8}\int_A \zeta^4(1-\rho^2)^2. \tag{10.9}$$

By Lemma 8.2 and Corollary 9.1, we have

$$\zeta^2\rho^2 = |v_\kappa|^2 \to 1 \quad \text{uniformly in } \overline{A} \text{ as } \kappa\to\infty. \tag{10.10}$$

By combining (10.9) and (10.10), we find that, for sufficiently large κ,

$$E_\kappa(v_\kappa) \ge E_\kappa(u_\kappa) + \frac{1}{2}\int_A \zeta^2|\nabla\rho|^2 + \frac{1}{3}\int_A |\nabla\psi|^2 + \frac{\kappa^2}{8}\int_A \zeta^4(1-\rho^2)^2. \tag{10.11}$$

Since both u_κ and v_κ are minimizers of (1.1)-(1.3), (10.11) implies that $\rho = 1$ and $\psi = c$ for some constant $c \in I\!R$. Taking the definitions of ρ and ψ into account, this in turn implies that $v_\kappa = \alpha u_\kappa$, where $\alpha = e^{\imath c} \in S^1$.

For further use, we also mention the following consequence of the method in [21]

Lemma 10.2. *Assume that $I_0 \le 2\pi$. Let $(v_\kappa) \subset \mathcal{J}$ be a family of solutions of the Ginzburg-Landau system*

$$\begin{cases} -\Delta v_\kappa &= \kappa^2 v_\kappa(1-|v_\kappa|^2) \quad \text{in } A\\ v_\kappa \wedge (\nu\cdot\nabla v_\kappa) &= 0 \quad \text{on } \partial A \end{cases} \tag{10.12}$$

such that $E_\kappa(v_\kappa) \le I_0$ and, up to subsequences, $v_\kappa \to u$ strongly in $H^1(A)$, where u is some minimizer of (1.14)-(1.15). Then there is some κ_0 such that, for $\kappa \ge \kappa_0$, v_κ is a minimizer of (1.1)-(1.3).

Proof : Let u_κ be a minimizer of (1.1)-(1.3). On the one hand, since $E_\kappa(v_\kappa) \le I_0$ and $v_\kappa \to u$ strongly in $H^1(A)$, we may obtain as in the proof of Lemma 8.2 that $|v_\kappa| \to 1$ uniformly in $\overline{A}$ as $\kappa\to\infty$. As explained in Section 9, the fact that $v_\kappa \to u$ in $H^1(A)$ together with the fact that $|v_\kappa| \to 1$ uniformly in $\overline{A}$ as $\kappa\to\infty$ imply that the conclusions of Corollary 9.1 apply to the family (v_κ). This allows us to proceed as in the proof of Proposition 10.1 in order to obtain (10.11). (The hypothesis (10.12) is needed in Lemma 10.1.) By reversing the roles of u_κ and v_κ, we thus also have, for sufficiently large κ,

$$E_\kappa(u_\kappa) \ge E_\kappa(v_\kappa) + \frac{1}{2}\int_A \zeta^2\rho^2\Big|\nabla\Big(\frac{1}{\rho}\Big)\Big|^2 + \frac{1}{3}\int_A |\nabla\psi|^2 + \frac{\kappa^2}{4}\int_A \zeta^4\rho^2\Big(1-\frac{1}{\rho^2}\Big)^2. \tag{10.13}$$

¿From (10.13) and the fact that $E_\kappa(u_\kappa) \le E_\kappa(v_\kappa)$, we find as above that $v_\kappa = \alpha_\kappa u_\kappa$, so that v_κ is a minimizer of (1.1)-(1.3).

10.2 Minimizers in presence of symmetries

The remaining part of this section is devoted to providing a more precise description of minimizers in presence of symmetries. To start with, we consider the case of a circular annulus. Assume that $A = \Omega \setminus \omega_0$, where $\Omega = \{z\,;|z| < R\}$, $\omega_0 = \{z\,;|z| < \rho\}$ and $\rho < R$. In this case, it is easy to see that (10.12) has a special solution, of the form $w_\kappa(z) = f_\kappa(|z|)\dfrac{z}{|z|} = f_\kappa(r)e^{\imath\theta}$. Moreover, this w_κ belongs to $\mathcal{J}$ and $f = f_\kappa$ satisfies

$$\begin{cases} -f'' - \dfrac{f'}{r} + \dfrac{f}{r^2} & = \kappa^2 f(1-f^2) \text{ in } [\rho, R] \\ f(\rho) = f(R) & = 1 \\ \dfrac{2\sqrt{R\rho}}{R+\rho} \le f \le 1 & \qquad\qquad \text{in } [\rho, R] \end{cases} \tag{10.14}$$

(see, e.g., [27]). This solution is obtained by minimizing E_κ in the class

$$\mathcal{L} = \{v \in H^1(A;\mathbb{C})\,;\, v(z) = g(|z|)\frac{z}{|z|},\ g \in H^1([\rho, R]),\ g(\rho) = g(R) = 1\}.$$

Since, by Lemma 2.4, one the minimizers of (1.14)-(1.15) is, for this A, $u_\infty(z) = \dfrac{z}{|z|} \in \mathcal{L}$, it follows that $E_\kappa(w_\kappa) \le I_0$.

Proposition 10.2. *Assume that $A = \Omega \setminus \omega_0$, where $\Omega = \{z\,;|z| < R\}$, $\omega_0 = \{z\,;|z| < \rho\}$ and $\rho < R$. Assume also that A is subcritical or critical, i.e., that $R/\rho \le e^2$. Then there is some κ_0 such that, for $\kappa \ge \kappa_0$, each minimizer u_κ of m_κ is of the form $u_\kappa(z) = \alpha_\kappa f_\kappa(|z|)\dfrac{z}{|z|}$ for some $\alpha_\kappa \in S^1$.*

Proof : We already noted that $E_\kappa(w_\kappa) \le I_0$. Therefore, in view of Lemma 10.2, it suffices to prove that $w_\kappa \to u_\infty$ in $H^1(A)$ as $\kappa \to \infty$. An easy computation shows that

$$\int_A (|\nabla(w_\kappa - u_\infty)|^2 + |w_\kappa - u_\infty|^2) = 2\pi \int_\rho^R \left((f-1)^2\left(r + \frac{1}{r}\right) + rf'^2\right) \le C \int_\rho^R ((f-1)^2 + f'^2), \tag{10.15}$$

where

$$C = 2\pi \max\{R + \frac{1}{R}, \rho + \frac{1}{\rho}, R\}. \tag{10.16}$$

If we multiply the first equation in (10.14) by $r(f-1)$, integrate and take the last property in (10.14) into account, we find

$$\int_\rho^R \left(\rho f'^2+\frac{2\kappa^2\sqrt{R\rho}}{R+\rho}\rho(f-1)^2\right) \le \int_\rho^R (rf'^2+\kappa^2 rf(f-1)^2(1+f)) = \int_\rho^R \frac{1}{r}f(1-f) \le \frac{1}{\rho}\int_\rho^R (1-f). \quad (10.17)$$

By Cauchy-Schwartz, we have

$$\frac{1}{\rho}\int_\rho^R (1-f) \le \frac{\kappa^2\sqrt{R\rho}}{R+\rho}\rho\int_\rho^R (f-1)^2+\frac{R+\rho}{4\kappa^2\rho^3\sqrt{R\rho}}\int_\rho^R 1. \quad (10.18)$$

Inserting (10.18) into (10.17), we are led to

$$\int_\rho^R \left(\rho f'^2+\frac{\kappa^2\sqrt{R\rho}}{R+\rho}\rho(f-1)^2\right) \le \frac{D}{\kappa^2}. \quad (10.19)$$

By combining (10.15) and (10.19), we find that $w_\kappa \to u_\infty$ in $H^1(A)$. The proof of Proposition 10.2 is complete.

We now turn to domains which are symmetric in the sense of Definition 2.3.

Proposition 10.3. *Assume that A is $\mathcal{O}$-symmetric. Assume also that A is subcritical or critical, i.e., that $I_0 \le 2\pi$.Then there is some κ_0 such that, if $\kappa \ge \kappa_0$, there is a minimizer v_κ of m_κ such that*

$$v_\kappa(\mathcal{O}(z)) = \mathcal{O}(v_\kappa(z)), \quad \forall\ z \in A. \quad (10.20)$$

Proof : Since circular annuli were treated in Proposition 10.2, we may assume that $\mathcal{O}$ is either a symmetry with respect to a line, or a rotation of angle $2\pi/n$ for some integer $n \ge 2$.

We start with the case of a symmetry, which we may assume with respect to Ox. Fix some $z_0 \in A\cap Ox$ and let (u_κ) be a family of minimizers of (1.1)-(1.3). For sufficiently large κ, we have $1/2 \le |u_\kappa| \le 1$, so that there is exactly one $\alpha_\kappa \in S^1$ such that $\alpha_\kappa u_\kappa(z_0) \in (0,1]$. Set $v_\kappa = \alpha_\kappa u_\kappa$. Let $w_\kappa(z) = \overline{v_\kappa(\overline{z})}$, which is clearly a minimizer of (1.1)-(1.3). Since $w_\kappa(z_0) = v_\kappa(z_0)$, Proposition 10.1 implies that $w_\kappa = v_\kappa$ for sufficiently large κ. Thus $\overline{v_\kappa(z)} = v_\kappa(\overline{z})$ in A, for sufficiently large κ, which is the desired conclusion.
Assume now that $\mathcal{O}$ is a rotation of angle $2\pi/n$, for example around the origin. Let (u_κ) be a family of minimizers of (1.1)-(1.3). As in the case of a symmetry, for sufficiently large κ, there is some $\alpha_\kappa \in S^1$ such that $u_\kappa \circ \mathcal{O} = \alpha_\kappa u_\kappa$. Since $u_\kappa \circ \mathcal{O}^n = u_\kappa$, we find that α_κ must be of the form $\alpha_\kappa = e^{\imath 2l\pi/n}$ for some integer l with $0 \le l \le n-1$. Fix some $z_0 \in \partial\Omega$ and let $z_k = \mathcal{O}^k(z_0)$, $k = 1, \dots, n$, so that $z_n = z_0$. Without any loss of generality, we may assume that $u_\kappa(z_0) = 1$. Denote by Γ_k the directly oriented arc of $\partial\Omega$ joining z_{k-1} to z_k. These arcs form a partition of $\partial\Omega$ and $\mathcal{O}(\Gamma_{k-1}) = \Gamma_k$

for $k = 1, \ldots, n-1$. Since each Γ_k is simply connected, we may write, on Γ_k, $u_\kappa(z) = e^{\imath\psi_k(z)}$ for some smooth ψ_k. Moreover, we may assume that $\psi_{k+1}(z_k) = \psi_k(z_k)$ for $k = 1, \ldots, n-1$, and that $\psi_1(z_0) = 0$. Using the fact that $u_\kappa(z_1) = \alpha_\kappa = e^{\imath 2l\pi/n}$, we find that $\psi_1(z_1) = 2l\pi/n + 2m\pi$ for some integer m, that is $\psi_2(\mathcal{O}(z_0)) = \psi_1(z_0) + 2l\pi/n + 2m\pi$. Since $u_\kappa \circ \mathcal{O} = \alpha_\kappa u_\kappa$, we must then have $\psi_2(\mathcal{O}(z)) = \psi_1(z) + 2l\pi/n + 2m\pi$ for all $z \in \Gamma_1$, by connectivity of Γ_1. Reiterating this argument, we find that, with the same m, we have $\psi_{k+1}(\mathcal{O}(z)) = \psi_k(z) + 2l\pi/n + 2m\pi$ for all $z \in \Gamma_k$, for $k = 1, \ldots, n-1$. Finally, since

$$1 = \deg(u_\kappa, \partial\Omega) = \frac{1}{2\pi}(\psi_n(z_n) - \psi_1(z_0)) = \frac{1}{2\pi}\sum_{k=1}^{n}(\psi_k(z_k) - \psi_k(z_{k-1})) = l + 2mn, \tag{10.21}$$

we find that $l = 1$ and $m = 0$. This implies that $u_\kappa \circ \mathcal{O} = e^{\imath 2\pi/n} u_\kappa$, that is $u_\kappa \circ \mathcal{O} = \mathcal{O} \circ u_\kappa$, which is the desired result.

Corollary 10.1. *Assume that, for some κ, m_κ is attained and that the minimizers of (1.1)-(1.3) are unique up to a phase shift. Assume also that A is $\mathcal{O}$-symmetric. Then :*
a) if $\mathcal{O}$ is a symmetry with respect to a straight line, there is a minimizer u_κ of m_κ such that $u_\kappa \circ \mathcal{O} = \mathcal{O} \circ u_\kappa$;
b) if $\mathcal{O}$ is a rotation, then all the minimizers of (1.1)-(1.3) satisfy $u_\kappa \circ \mathcal{O} = \mathcal{O} \circ u_\kappa$;
c) if A is a circular annulus, $A = \{z\,;\,\rho < |z| < R\}$, then all the minimizers of (1.1)-(1.3) are radially symmetric, i.e., of the form $u_\kappa(z) = f(|z|)\dfrac{z}{|z|}$.

Remark 10.1. In the above statement, we do not assume A subcritical or critical.

Proof of Corollary 10.1 : When $\mathcal{O}$ is a symmetry, property a) was obtained in Proposition 10.3 using two ingredients : uniqueness up to a phase shift and the existence of a point $z_0 \in A$ such that $|u_\kappa(z_0)| \geq 1/2$. However, such a point exists each time m_κ is attained. Indeed, recall that, by Lemma 4.4, we have $u_\kappa \in C^\infty(\overline{A})$, and the existence of z_0 follows from the fact that $|u_\kappa| = 1$ on ∂A. Similarly, for $\delta > 0$ sufficiently small, we have $|u_\kappa| \geq 1/2$ on Γ, where $\Gamma = \{z \in A\,;\,\mathrm{dist}(z, \partial\Omega) = \delta\}$. If $\mathcal{O}$ is a rotation of angle $2\pi/n$, then this Γ is $\mathcal{O}$-symmetric. We may consider this Γ in the proof of Proposition 10.3 and obtain symmetry of minimizers. Finally, in case c), minimizers are symmetric with respect to rotations of angle $2\pi/n$, for all n, and the conclusion follows by density of rational rotations among all the rotations.

11 Asymptotic behavior of the quasi-minimizers in the supercritical case $I_0 > 2\pi$

11.1 Concentration of the energy and of the zeroes near $\partial\Omega \cup \partial\omega_0$

Throughout this section, we assume that $I_0 > 2\pi$. We consider a family (u_κ) of quasi-minimizers in the sense of the Definition 7.1. The purpose of this section is to give a precise description of the quasi-minimizers. Roughly speaking, we are going to prove that "everything happens near $\partial\Omega$ and near $\partial\omega_0$" ; rigorous statements will be given below.

To start with, we are going to prove that, for large κ, almost all the energy of u_κ is concentrated near the two distinguished parts of the boundary, $\partial\Omega$ and $\partial\omega_0$.

Lemma 11.1. *Let K be a fixed compact in $\overline{A}\setminus(\partial\Omega\cup\partial\omega_0)$. Then, for any $m\in I\!N$, we have*

$$\lim_{\kappa\to\infty}\kappa^m\Big(\kappa^2\int_A(1-|u_\kappa|^2)^2+\int_K|\nabla u_\kappa|^2\Big)=0. \tag{11.1}$$

Proof : We argue by induction, starting with $m=2$. Fix some compact $K\subset\overline{A}\setminus(\partial\Omega\cup\partial\omega_0)$. Let, for $\delta>0$ fixed and sufficiently small, $\Gamma=\Gamma_\delta=\{z\in A\,;\,\mathrm{dist}(z,\partial\Omega)=\delta\}$ and $\gamma=\gamma_\delta=\{z\in A\,;\,\mathrm{dist}(z,\partial\omega_0)=\delta\}$. Let $U=U_\delta$ be the domain enclosed by $\partial\Omega$ and Γ, $V=V_\delta$ be the domain enclosed by $\partial\omega_0$ and γ and set $W=W_\delta=\overline{A}\setminus(\overline{U}\cup\overline{V})$. Then

$$K\subset W,\ \partial U=\partial\Omega\cup\Gamma,\ \partial V=\partial\omega_0\cup\gamma, \tag{11.2}$$

provided δ is sufficiently small. Following the argument that led us to the inequality (9.13) in the proof of Lemma 9.2, we find successively for sufficiently large κ, that

$$E_\kappa(u_\kappa)\geq\frac12\int_U|\nabla u_\kappa|^2+\frac12\int_V|\nabla u_\kappa|^2+\frac12\int_K|\nabla u_\kappa|^2+\frac{\kappa^2}{4}\int_A(1-|u_\kappa|^2)^2, \tag{11.3}$$

then

$$E_\kappa(u_\kappa)\geq\frac12\int_K|\nabla u_\kappa|^2+\frac{\kappa^2}{4}\int_A(1-|u_\kappa|^2)^2+2\pi-\int_\Gamma(1-|u_\kappa|^2)|\nabla u_\kappa|-\int_\gamma(1-|u_\kappa|^2)|\nabla u_\kappa|, \tag{11.4}$$

and finally that

$$E_\kappa(u_\kappa)\geq\frac12\int_K|\nabla u_\kappa|^2+\frac{\kappa^2}{4}\int_A(1-|u_\kappa|^2)^2+2\pi-o\Big(\frac{1}{\kappa^2}\Big)\ \text{ as }\kappa\to\infty. \tag{11.5}$$

By (6.4) and (7.1), we have on the other hand that

$$E_\kappa(u_\kappa)\leq 2\pi+\frac{1}{e^\kappa}. \tag{11.6}$$

Estimate (11.1) for $m=2$ follows from (11.5) and (11.6).

We next assume that (11.1) holds for some $m\geq 2$ and all compacts K and establish (11.1) for $m+1$ and all compacts K, which will complete the proof. Fix a compact $K\subset\overline{A}\setminus(\partial\Omega\cup\partial\omega_0)$.

Fix some sufficiently small $0 < \delta_1 < \delta_2$ such that , for $\delta_1 < \delta < \delta_2$, (11.2) holds. By (11.1) applied to m and to the compact $L = (\overline{U_{\delta_2}} \setminus U_{\delta_1}) \cup (\overline{V_{\delta_2}} \setminus V_{\delta_1})$, we find that

$$\lim_{\kappa\to\infty} \kappa^m \Big(\kappa^2 \int_L (1-|u_\kappa|^2)^2 + \int_L |\nabla u_\kappa|^2 \Big) = 0. \tag{11.7}$$

By Fubini and (11.7), there is some $\delta = \delta^\kappa \in (\delta_1, \delta_2)$ such that

$$\lim_{\kappa\to\infty} \kappa^m \Big(\kappa^2 \int_{\Gamma_\delta\cup\gamma_\delta} (1-|u_\kappa|^2)^2 + \int_{\Gamma_\delta\cup\gamma_\delta} |\nabla u_\kappa|^2 \Big) = 0. \tag{11.8}$$

Thus

$$\int_{\Gamma_\delta\cup\gamma_\delta} (1-|u_\kappa|^2)|\nabla u_\kappa| \le \Big(\int_{\Gamma_\delta\cup\gamma_\delta} (1-|u_\kappa|^2)^2 \int_{\Gamma_\delta\cup\gamma_\delta} |\nabla u_\kappa|^2 \Big)^{1/2} = o\Big(\frac{1}{\kappa^{m+1}}\Big) \text{ as } \kappa\to\infty. \tag{11.9}$$

By combining (11.4), (11.6) and (11.9), we obtain (11.1) for $m+1$ and K.

Corollary 11.1. *Assume that $I_0 > 2\pi$. Then*

$$2\pi - o\Big(\frac{1}{\kappa^m}\Big) \le m_\kappa \le 2\pi \quad \text{as } \kappa\to\infty, \quad \text{for } m \in I\!N. \tag{11.10}$$

We next prove that the zeroes of u_κ are very close to $\partial\Omega\cup\partial\omega_0$. For this purpose, we define the sets

$$A_m^\kappa = \Big\{ z \in \overline{A}\,;\ \mathrm{dist}(z,\, \partial\Omega\cup\partial\omega_0) \ge \frac{1}{\kappa^m} \Big\}, \quad \text{for } m \in I\!N. \tag{11.11}$$

By Lemma C.1, u_κ has to vanish near $\partial\Omega$ and near $\partial\omega_0$. The next results implies in particular that, for each fixed $m \in I\!N$, the zeroes of u_κ are at distance $o\Big(\dfrac{1}{\kappa^m}\Big)$ from $\partial\Omega\cup\partial\omega_0$ as $\kappa\to\infty$:

Lemma 11.2. *We have*

$$\lim_{\kappa\to\infty} \inf_{A_m^\kappa} |u_\kappa| = 1, \quad \text{for } m \in I\!N. \tag{11.12}$$

Proof : Argue by contradiction and assume that, for some $m \in I\!N$ and some $t \in (0,1)$, we may find sequences $\kappa_l \to \infty$, $(z_l) \subset A_m^{\kappa_l}$ such that $|u_{\kappa_l}(z_l)| \le t$. We first claim that the sequence (z_l) stays far away from $\cup_{j=1}^k \partial\omega_j$. Indeed, let $U \subset A$ be a smooth open set such that

$$\cup_{j=1}^k \partial\omega_j \subset \overline{U} \subset \overline{A} \setminus (\partial\Omega\cup\partial\omega_0). \tag{11.13}$$

By (11.1), we have

$$\lim_{\kappa\to\infty} \int_U |\nabla u_\kappa|^2 = 0. \tag{11.14}$$

On the other hand, by Proposition 7.1, we may find a family $(\alpha_\kappa) \subset S^1$ such that

$$\alpha_\kappa u_\kappa \rightharpoonup 1 \quad \text{weakly in } H^1(A) \text{ as } \kappa \to \infty. \tag{11.15}$$

By (11.14) and (11.15), we find that

$$v_\kappa = \alpha_\kappa u_\kappa \to 1 \quad \text{strongly in } H^1(U) \text{ as } \kappa \to \infty. \tag{11.16}$$

Since it suffices to prove (11.12) for v_κ, we work from now on with v_κ instead of u_κ (note that v_κ is also a quasi-minimizer). Set $g = g_\kappa = \mathrm{tr}_{\cup_{j=1}^k \partial\omega_j} v_\kappa$, so that $g \to 1$ in $H^{1/2}(\cup_{j=1}^k \partial\omega_j)$. Set also $h = h_\kappa = \mathrm{tr}_{\partial\Omega\cup\partial\omega_0} v_\kappa - 1$. We split $v_\kappa = a + b + c = a_\kappa + b_\kappa + c_\kappa$, where a, b, c satisfy respectively :

$$\begin{cases} \Delta a &= 0 \quad \text{in } A \\ a &= g \quad \text{on } \cup_{j=1}^k \partial\omega_j \\ a &= 1 \quad \text{on } \partial\Omega \cup \partial\omega_0 \end{cases}, \tag{11.17}$$

$$\begin{cases} \Delta b &= 0 \quad \text{in } A \\ b &= 0 \quad \text{on } \cup_{j=1}^k \partial\omega_j \\ b &= h \quad \text{on } \partial\Omega \cup \partial\omega_0 \end{cases} \tag{11.18}$$

and

$$\begin{cases} -\Delta c &= \kappa^2 v_\kappa (1 - |v_\kappa|^2) \quad \text{in } A \\ c &= 0 \quad \text{on } \partial A \end{cases}. \tag{11.19}$$

By Lemma 8.3, there is some $\delta > 0$ independent of sufficiently large l such that

$$|a(z)| \geq \frac{1+t}{2} \quad \text{if } \mathrm{dist}(z, \partial A) \leq \delta. \tag{11.20}$$

On the other hand, since v_κ is bounded in $H^1(A)$, it follows that h is bounded in $H^{1/2}(\partial\Omega \cup \partial\omega_0)$. By standard elliptic estimates ([26]), we find that

$$|\nabla b(z)| \leq C \quad \text{for } z \in \overline{U}, \tag{11.21}$$

for some constant C independent of l. Since $b = 0$ on $\cup_{j=1}^k \partial\omega_j$, this implies that

$$|b(z)| \leq C \ \mathrm{dist}(z, \cup_{j=1}^k \partial\omega_j) \quad \text{for } z \in U. \tag{11.22}$$

Finally, by Lemma 11.1 we have $\|\Delta c\|_{L^p(A)} \to 0$ as $l \to \infty$, so that $\|c\|_{W^{2,p}(A)} \to 0$ as $l \to \infty$, for $1 < p < \infty$. In particular, by the Sobolev embeddings and the fact that $c = 0$ on ∂A, we have

$$|c(z)| + |\nabla c(z)| \to 0 \quad \text{uniformly in } \overline{A} \text{ as } l \to \infty. \tag{11.23}$$

By combining (11.20), (11.22) and (11.23), we find that there is some $\varepsilon > 0$ independent of sufficiently large l such that

$$|v_\kappa(z)| > t \quad \text{if } \mathrm{dist}(z, \cup_{j=1}^k \partial\omega_j) \leq \varepsilon, \tag{11.24}$$

that is, as claimed, the points z_l are far away from $\cup_{j=1}^k \partial\omega_j$.

We next prove that the points z_l are close to $\partial\Omega \cup \partial\omega_0$. Indeed, since $|v_{\kappa_l}(z_l)| \le t$, we find from Lemma 7.1 that

$$\operatorname{dist}(z_l, \partial A) \to 0 \quad \text{ as } l \to \infty. \tag{11.25}$$

By (11.24) and (11.25) and the hypothesis $z_l \in A_m^{\kappa_l}$, we obtain that

$$\frac{1}{\kappa_l^m} \le d = d_l = \operatorname{dist}(z_l,\, \partial\Omega \cup \partial\omega_0) \to 0 \quad \text{ as } l \to \infty. \tag{11.26}$$

Finally, we will see that the existence of the points z_l contradicts the conclusion of Lemma 11.1. We start from the fact that $a+b$ satisfies

$$\begin{cases} \Delta(a+b) &= 0 \quad \text{in } A \\ \quad |a+b| &= 1 \quad \text{on } \partial A \end{cases}, \tag{11.27}$$

so that

$$|\nabla(a+b)(z)| \le \frac{C}{\operatorname{dist}(z, \partial A)} \quad \text{ for } z \in A, \tag{11.28}$$

for some constant C independent of z and l, by standard estimates for the Green function. By combining (11.23), (11.26) and (11.28), we see that for sufficiently large l and a constant D independent of l we have

$$B = B_l = \{z\,;\, |z - z_l| < d_l/2\} \subset A \quad \text{ and } \quad |\nabla v_{\kappa_l}(z)| \le \frac{D}{d_l} \text{ in } B. \tag{11.29}$$

By (11.29) and the hypothesis $|v_{\kappa_l}(z_l)| \le t$, we find that there is some $0 < c < 1/2$ independent of large l such that

$$|v_{\kappa_l}(z)| \le \frac{1+t}{2} \quad \text{ for } z \text{ such that } |z - z_l| < c\, d_l. \tag{11.30}$$

By (11.29) and (11.30), we find

$$\kappa_l^2 \int_A (1 - |v_{\kappa_l}|^2)^2 \ge \kappa_l^2 \int_{\{z\,;|z-z_l|<c\,d_l\}} (1 - |v_{\kappa_l}|^2)^2 \ge E\kappa_l^{2+2m}, \tag{11.31}$$

for some $E > 0$ independent of large l. This contradicts Lemma 11.1, and thus completes the proof of Lemma 11.2.

As a final step towards a sharp description of the quasi-minimizers, we prove that, in compact subsets of A, u_κ is very close to a constant of modulus 1.

Lemma 11.3. *There are constants $\beta_\kappa \in S^1$ such that, for any fixed compact $K \subset A$ and any integers $l \in I\!N$, $m \in I\!N$, we have*

$$\|\beta_\kappa - u_\kappa\|_{C^l(K)} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{as } \kappa \to \infty. \tag{11.32}$$

Proof : For the convenience of the reader, we divide the proof into several steps.

Step 1. Estimates for the C^1 convergence of the phase

Fix any point $z_0 \in A$ and set, for large κ, $\beta_\kappa = \dfrac{u_\kappa(z_0)}{|u_\kappa(z_0)|} \in S^1$; we are going to prove that (11.32) is satisfied for this choice of β_κ. Let a_κ be such that $\beta_\kappa = e^{\imath a_\kappa}$. Fix a compact $K \subset A$. In view of the statement we want to prove, we may assume that $z_0 \in K$ and that K is simply connected. Fix also a smooth simply connected open set U such that $K \subset U \subset \overline{U} \subset A$. In view of Proposition 7.1, for sufficiently large κ we have $|u_\kappa| \geq 1/2$ in $\overline{U}$, and thus we may write, globally in $\overline{U}$, $u_\kappa = \rho e^{\imath\psi} = \rho_\kappa e^{\imath\psi_\kappa}$, for some smooth ρ and ψ. We may always assume that $\psi_\kappa(z_0) = a_\kappa$. Set, for $\delta > 0$ sufficiently small, $\Gamma_\delta = \{z \in U\,;\, \mathrm{dist}(z, \partial U)\} = \delta\}$. By Fubini and Lemma 11.1, for any fixed $m \in I\!N$ we may find a $\delta = \delta(m, \kappa)$ such that

$$\Gamma_\delta \text{ encloses } K \quad \text{and} \quad \int_{\Gamma_\delta} (\kappa^2(1-|u_\kappa|^2)^2 + |\nabla u_\kappa|^2) = o\Big(\frac{1}{\kappa^{2m}}\Big) \quad \text{as } \kappa \to \infty. \tag{11.33}$$

Since $|\nabla\psi| \leq 2|\nabla u_\kappa|$, it follows in particular that

$$\int_{\Gamma_\delta} |\nabla\psi|^2 = o\Big(\frac{1}{\kappa^{2m}}\Big) \quad \text{as } \kappa \to \infty, \tag{11.34}$$

so that

$$\max\{\psi(z)\,;\, z \in \Gamma_\delta\} - \min\{\psi(z)\,;\, z \in \Gamma_\delta\} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{as } \kappa \to \infty. \tag{11.35}$$

Recall that, by (4.17), ψ satisfies

$$\mathrm{div}(\rho^2\nabla\psi) = 0 \quad \text{in } U. \tag{11.36}$$

By (11.35) and the fact that $\psi_\kappa(z_0) = a_\kappa$, we find that

$$\max\{|\psi(z) - a_\kappa|\,;\, z \in \Gamma_\delta\} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{as } \kappa \to \infty, \tag{11.37}$$

so that the maximum principle applied to ψ yields

$$\max\{|e^{\imath\psi(z)} - \beta_\kappa|\,;\, z \in K\} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{as } \kappa \to \infty. \tag{11.38}$$

We next estimate the rate of convergence of ψ to a_κ in C^1. To this purpose, we rewrite (11.36) as

$$-\Delta\psi = -\Delta(\psi - a_\kappa) = f = f_\kappa = \frac{2}{\rho}\nabla\rho\cdot\nabla\psi \quad \text{in } U. \tag{11.39}$$

Thus, by standard elliptic interior estimates, we have, for $p > 2$,

$$\|\nabla\psi\|_{L^\infty(K)} \le C_p(\|f\|_{L^p(U)} + \|\psi - a_\kappa\|_{L^\infty(\Gamma_\delta)}) = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{as } \kappa \to \infty. \tag{11.40}$$

Here, we use the fact that $|f| \le 8|\nabla u_\kappa|^2$ in U, together with Lemma 11.1 and (11.35).

Step 2. Estimates for the uniform convergence of the modulus
By (4.16), the equation of $\zeta = \zeta_\kappa = 1 - \rho = 1 - \rho_\kappa$ is

$$\Delta\zeta = g = g_\kappa = \kappa^2\rho(1-\rho^2) - \rho|\nabla\psi|^2 \quad \text{in } U. \tag{11.41}$$

By Lemma 11.1 and Step 1 applied to the compact $\overline{U}$, we have

$$\|g\|_{L^2(U)} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{as } \kappa \to \infty. \tag{11.42}$$

On the other hand, the inequality $|\nabla\rho| \le |\nabla u_\kappa|$ together with (11.33) imply that

$$\int_{\Gamma_\delta} (\kappa^2(1-\rho^2)^2 + |\nabla\rho|^2) = o\Big(\frac{1}{\kappa^{2m}}\Big) \quad \text{as } \kappa \to \infty, \tag{11.43}$$

which in turn yields

$$\|\zeta\|_{L^\infty(\Gamma_\delta)} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{as } \kappa \to \infty. \tag{11.44}$$

By (11.42), (11.44) and standard elliptic interior estimates, we obtain

$$\|1-\rho\|_{L^\infty(K)} = \|\zeta\|_{L^\infty(K)} \le C(\|g\|_{L^2(U)} + \|\zeta\|_{L^\infty(\Gamma_\delta)}) = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{as } \kappa \to \infty \tag{11.45}$$

and, for $1 \le p < \infty$,

$$\|\nabla\rho\|_{L^p(K)} = \|\nabla\zeta\|_{L^p(K)} \le C_p(\|g\|_{L^2(U)} + \|\zeta\|_{L^\infty(\Gamma_\delta)}) = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{as } \kappa \to \infty. \tag{11.46}$$

By combining (11.38), (11.40), (11.45) and (11.46), we find that

$$\|u_\kappa - \beta_\kappa\|_{W^{1,p}(K)} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{as } \kappa \to \infty,\ \forall\, K, \forall\, 1 \le p < \infty. \tag{11.47}$$

Step 3. The bootstrap argument

Let f be as in (11.39). By (11.40) and (11.46) applied to the compact $\overline{U}$, we have

$$\|f\|_{L^p(U)} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{ as } \kappa \to \infty, \forall\, 1 < p < \infty. \tag{11.48}$$

By combining (11.37), (11.39) and (11.48) we find, using standard elliptic interior estimates, that

$$\|\psi - a_\kappa\|_{W^{2,p}(K)} \leq C_p(\|f\|_{L^p(U)} + \|\psi - a_\kappa\|_{L^\infty(\Gamma_\delta)}) = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{ as } \kappa \to \infty, \forall\, 1 < p < \infty, \tag{11.49}$$

so that

$$\|\psi - a_\kappa\|_{C^{1,a}(K)} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{ as } \kappa \to \infty, \forall\, 0 < a < 1, \tag{11.50}$$

by the Sobolev embeddings.
Let now g be as in (11.41). By (11.40) and (11.45) applied to the compact $\overline{U}$, we find that

$$\|g\|_{L^p(U)} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{ as } \kappa \to \infty, \forall\, 1 < p < \infty. \tag{11.51}$$

From (11.41), (11.44) and (11.51) we obtain, using standard elliptic interior estimates, that

$$\|1 - \rho\|_{W^{2,p}(K)} \leq C_p(\|g\|_{L^p(U)} + \|1 - \rho\|_{L^\infty(\Gamma_\delta)}) = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{ as } \kappa \to \infty, \forall\, 1 < p < \infty, \tag{11.52}$$

and thus

$$\|1 - \rho\|_{C^{1,a}(K)} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{ as } \kappa \to \infty, \forall\, 0 < a < 1, \tag{11.53}$$

using once again the Sobolev embeddings.
By combining (11.50) and (11.53), we are led to

$$\|u_\kappa - \beta_\kappa\|_{C^{1,a}(K)} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{ as } \kappa \to \infty, \forall\, 0 < a < 1. \tag{11.54}$$

We complete the proof of Lemma 11.3 by establishing by a straightforward induction the estimate

$$\|u_\kappa - \beta_\kappa\|_{C^{l,a}(K)} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{ as } \kappa \to \infty, \forall\, 0 < a < 1, \forall\, l \in \mathbb{N}. \tag{11.55}$$

Assuming that (11.55) holds for l and all the compact subsets of A, we find, with f and g given respectively by (11.39) and (11.41), that

$$\|f\|_{C^{l-1,a}(U)} = o\Big(\frac{1}{\kappa^m}\Big), \quad \|g\|_{C^{l-1,a}(U)} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{ as } \kappa \to \infty, \forall\, 0 < a < 1. \tag{11.56}$$

We next replace (11.50) and (11.52) by the appropriate Schauder interior estimates

$$\|\psi - a_\kappa\|_{C^{l+1,a}(K)} \leq C_a(\|f\|_{C^{l-1,a}(U)} + \|\psi - a_\kappa\|_{L^\infty(\Gamma_\delta)}) = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{ as } \kappa \to \infty, \forall\, 0 < a < 1, \tag{11.57}$$

and respectively

$$\|1-\rho\|_{C^{l+1,a}(K)} \leq C_a(\|g\|_{C^{l-1,a}(U)} + \|1-\rho\|_{L^\infty(\Gamma_\delta)}) = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{as } \kappa \to \infty, \forall\, 0 < a < 1. \tag{11.58}$$

It follows from (11.57) and (11.58) that

$$\|u_\kappa - \beta_\kappa\|_{C^{l+1,a}(K)} = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{as } \kappa \to \infty, \forall\, 0 < a < 1. \tag{11.59}$$

The proof of Lemma 11.3 is complete.

11.2 The profile of quasi-minimizers for large κ

In order to better explain the results we will establish in the remaining part of this section, we start by taking a closer look to the upper bound $m_\kappa \leq 2\pi$ established in Proposition 6.1. More specifically, we will give a more intuitive description of the testing maps $w_n = u_n v_n$ constructed during the proof of Proposition 6.1. To start with, let us recall the construction that led to u_n. Fix a conformal representation F of $\overline{\Omega}$ into $\overline{I\!D}$ (this map was denoted w in the proof of Lemma 6.1) and let, for $a \in I\!D$, F_a to be the conformal representation of $\overline{\Omega}$ into $\overline{I\!D}$ given by $F_a = u_a \circ F$, where u_a is the Moebius map, $u_a(z) = \dfrac{z-a}{1-\overline{a}z}$. Thus F_a is holomorphic and has exactly one zero $z_a = F^{-1}(a)$. In particular, this zero tends to $\partial\Omega$ as $|a| \nearrow 1$; more precisely, $\mathrm{dist}(z_a, \partial\Omega)$ is of the order of $1-|a|$. Another easily seen property of F_a is that $F_a \to -1$ uniformly on compact subsets of Ω as $|a| \nearrow 1$; more specifically, on any compact $K \subset \Omega$, the quantity $|F_a + 1|$ is of the order of $1-|a|$. Another useful remark is that the map F_a is determined, up to a phase shift, by its zero z_a : if H is a conformal representation of $\overline{\Omega}$ into $\overline{I\!D}$ such that $H(z_a) = 0$, then there is some $\alpha \in S^1$ such that $H = \alpha F_a$ (see, e.g., [1]). The maps u_n constructed in the proof of Lemma 6.1 were essentially given by $u_n \approx F_{a_n}$ for some sequence $(a_n) \subset (0,1)$ such that $a_n \nearrow 1$. Here, $\approx$ accounts for the fact that the modulus of F_{a_n} was slightly corrected in order to have $|u_n| = 1$ on ∂A ; however, this correction is less and less important as $n \to \infty$. We also note that, for the purpose of Lemma 6.1, any sequence (a_n) such that $|a_n| \to 1$ would have been useful. For later use, it will be convenient to describe the map F_a not in terms of the parameter a in the Moebius transform, but rather in terms of its unique zero. For a given $z \in \Omega$, we will denote F^z the map $F_{F(z)}$, which is the only map of the form F_a that vanishes at z.
We next recall the construction of v_n. Let G be a conformal representation of $\mathbb{C} \cup \{\infty\} \setminus \overline{\omega_0}$ into $I\!D$; it is easy to see that G extends as a smooth map from $\partial\omega_0$ into S^1. Set, as above, $G_a = u_a \circ G$. It is a simple exercice that the map v_n constructed in the proof of Lemma 6.2 essentially agrees with some $\overline{G_a}$: $v_n \approx \overline{G_{b_n}}$, provided we choose the right G, for some sequence $(b_n) \subset (0,1)$ such that $b_n \nearrow 1$. Similarly, we define $G^z = G_{G(z)}$, for $z \in \overline{\mathbb{C}} \setminus \overline{\omega_0}$.
Finally, the testing maps in Proposition 6.1 are given by

$$w_n \approx F_{a_n}\overline{G_{b_n}} = F^{\zeta_n}\overline{G^{\xi_n}}, \qquad \text{where } \zeta_n = F^{-1}(a_n),\, \xi_n = G^{-1}(b_n). \tag{11.60}$$

In particular, w_n has exactly two zeroes : ζ_n near $\partial\Omega$, ξ_n near $\partial\omega_0$. Moreover, intuitively speaking, w_n is "almost" holomorphic far away from $\partial\omega_0$ (since $v_n \approx -1$, so that $w_n \approx -u_n$ there) and, similarly, "almost" anti-holomorphic far away from $\partial\Omega$. A straightforward adaptation of the proof of Proposition 6.1 yields the following result, whose proof will be omitted

Lemma 11.4. *Let F be a conformal representation of $\overline{\Omega}$ into $\overline{I\!D}$ and G be a conformal representation of $\overline{C} \setminus \omega_0$ into $\overline{I\!D}$. Define, for $\zeta, \xi \in A$ and $\alpha \in S^1$, $w_{\zeta,\xi,\alpha} = \alpha F^\zeta \overline{G^\xi}$, $w_{\zeta,\xi,\alpha} : \overline{A} \to C$. Then :*
a) $w_{\zeta,\xi,\alpha} \to \alpha$ uniformly on the compacts of $\overline{A} \setminus (\partial\Omega \cup \partial\omega_0)$ as $\zeta \to \partial\Omega$ and $\xi \to \partial\omega_0$;
b) $\displaystyle\lim_{\zeta\to\partial\Omega,\,\xi\to\partial\omega_0} E_\kappa(w_{\zeta,\xi,\alpha}) = 2\pi$.

Since, in principle we can have $m_\kappa < 2\pi$, we can not infer that, for a fixed κ, $w_{\zeta,\xi,\alpha}$ satisfies the upper bound (7.1) required in the definition of a quasi-minimizer, even if we choose ζ, ξ very close to $\partial\Omega$, to $\partial\omega_0$ respectively. However, if we assume A supercritical, we know from Corollary 6.6 that $m_\kappa \approx 2\pi$ for large values of κ, so that, for large κ, $w_{\zeta,\xi,\alpha}$ becomes a good candidate for a quasi-minimizer, provided we take ζ and ξ sufficiently close to $\partial\Omega$, respectively to $\partial\omega_0$. Unfortunately, this is unrealistic. Indeed, the above construction can be modified in the following way, which we present informally : take a map that is close to αF^ζ near ζ and close to $\beta\overline{G^\xi}$ near ξ. Here, $\alpha \neq \beta$, $\alpha, \beta \in S^1$. Then "glue" these maps by considering an S^1-valued transition map from ζ to ξ, that is equal to $-\alpha$ near ζ and to $-\beta$ near ξ. If the transition map is properly choosen, then we obtain a map with energy close to 2π. Therefore, we may prove at best that, near its zeroes, a quasi-minimizer looks like some αF^ζ (if the zero is close to $\partial\Omega$), respectively like some $\beta\overline{G^\xi}$ (near $\partial\omega_0$). We will eventually prove that this is indeed the case for large κ.

We state informally the main results of this section : assume A supercritical. Then, for sufficiently large κ, the following properties hold :
(P1) each quasi-minimizer u_κ has exactly two zeroes, one, say ζ, near $\partial\Omega$, the other one, say ξ, near $\partial\omega_0$;
(P2) each quasi-minimizer u_κ is "almost" holomorphic far away from $\partial\omega_0$ and "almost" anti-holomorphic far away from $\partial\Omega$;
(P3) near ζ, u_κ is close (in a sense to be made precise later) to αF^ζ for some $\alpha \in S^1$. Similarly, near ξ, u_κ is close to some $\beta\overline{G^\xi}$;
(P4) ζ is a zero of degree 1 and ξ is a zero of degree -1.

We start by giving a precise version of property (P2). Recall that f is holomorphic if and only if $|\nabla f|^2 = 2\text{Jac}\ f$; for a general map f, we have only the pointwise inequality $|\nabla f|^2 \geq 2\text{Jac}\ f$. Similarly, g is anti-holomorphic if and only if $|\nabla g|^2 = -2\text{Jac}\ g$, while in general we have only $|\nabla g|^2 \geq -2\text{Jac}\ g$.

Lemma 11.5. *Assume that $I_0 > 2\pi$ and let (u_κ) be a family of quasi-minimizers. Let K be a* *fixed compact set in $\overline{A}\setminus\partial\omega_0$, L be a fixed compact set in $\overline{A}\setminus\partial\Omega$. Then, for each $m\in I\!N$, we have*

$$\int\limits_K (|\nabla u_\kappa|^2 - 2\mathrm{Jac}\ u_\kappa) = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{and} \quad \int\limits_L (|\nabla u_\kappa|^2 + 2\mathrm{Jac}\ u_\kappa) = o\Big(\frac{1}{\kappa^m}\Big) \quad \text{as } \kappa\to\infty. \tag{11.61}$$

Proof : By Lemma 11.3, the conclusion is clear if either K or L are compact subsets of $\overline{A}\setminus(\partial\Omega\cup\partial\omega_0)$. Therefore, without any loss of generality, we may assume that $K=\overline{U}$, $L=\overline{V}$, where $U,V\subset A$ are smooth open sets such that

$$\partial\Omega\subset\overline{U}\subset\overline{A}\setminus(\cup_{j=0}^k\partial\omega_j), \quad \partial\omega_0\subset\overline{V}\subset\overline{A}\setminus(\partial\Omega\cup\cup_{j=1}^k\partial\omega_j), \quad \overline{U}\cap\overline{V}=\emptyset. \tag{11.62}$$

Set $\Gamma=\partial U\setminus\partial\Omega$, $\gamma=\partial V\setminus\partial\omega_0$, which are smooth (possibly multiply connected) curves in A. We proceed as in the proof of Lemma 11.1. We have

$$E_\kappa(u_\kappa)\ge\int\limits_K\Big(\frac12|\nabla u_\kappa|^2-\mathrm{Jac}\ u_\kappa\Big)+\int\limits_K\mathrm{Jac}\ u_\kappa+\int\limits_L\Big(\frac12|\nabla u_\kappa|^2+\mathrm{Jac}\ u_\kappa\Big)-\int\limits_L\mathrm{Jac}\ u_\kappa. \tag{11.63}$$

Since, by Lemma 2.1, the degree formula (1.5) and Lemma 11.3, we have

$$\int\limits_K\mathrm{Jac}\ u_\kappa=\frac12\int\limits_{\partial\Omega}u_\kappa\wedge\frac{\partial u_\kappa}{\partial\tau}+\frac12\int\limits_\Gamma u_\kappa\wedge\frac{\partial u_\kappa}{\partial\tau}=\pi+o\Big(\frac{1}{\kappa^m}\Big)\quad \text{as } \kappa\to\infty \tag{11.64}$$

and similarly

$$-\int\limits_L\mathrm{Jac}\ u_\kappa=-\frac12\int\limits_{\partial\omega_0}u_\kappa\wedge\frac{\partial u_\kappa}{\partial\tau}-\frac12\int\limits_\gamma u_\kappa\wedge\frac{\partial u_\kappa}{\partial\tau}=\pi+o\Big(\frac{1}{\kappa^m}\Big)\quad \text{as } \kappa\to\infty, \tag{11.65}$$

the conclusion of Lemma 11.5 follows by combining (11.63), (11.64) and (11.65) together with the upper bound

$$E_\kappa(u_\kappa)\le m_\kappa+\frac{1}{e^\kappa}\le 2\pi+\frac{1}{e^\kappa} \tag{11.66}$$

required in the definition of quasi-minimizers.

If we examine the above proof, the argument we used yields in a straightforward way the following result, whose proof will be omitted

Lemma 11.6. *Let U, $V\subset A$ be two smooth open sets such that (11.62) holds. Then, for each $m\in I\!N$, we have*

$$\int\limits_U\Big(\frac12|\nabla u_\kappa|^2+\frac{\kappa^2}{4}(1-|u_\kappa|^2)^2\Big)=\pi+o\Big(\frac{1}{\kappa^m}\Big),\ \int\limits_V\Big(\frac12|\nabla u_\kappa|^2+\frac{\kappa^2}{4}(1-|u_\kappa|^2)^2\Big)=\pi+o\Big(\frac{1}{\kappa^m}\Big),\ \text{as } \kappa\to\infty. \tag{11.67}$$

We next start preparing the proof of property (P3). To this purpose, it will be convenient to approximate u_κ with a product, one of the factors "living" near $\partial\Omega$, the other one near $\partial\omega_0$. This is done in the following way : set $v_\kappa = \overline{\beta_\kappa} u_\kappa$, where β_κ is given by Lemma 11.3. Fix two small numbers $0 < \delta_1 < \delta_2$ and set

$$U_j = \{z \in A\,;\, \mathrm{dist}(z, \partial\Omega\} < \delta_j\}, \quad V_j = \{z \in A\,;\, \mathrm{dist}(z, \partial\Omega\} < \delta_j\}, \quad j = 1, 2. \tag{11.68}$$

By Lemma 11.3, we may extend $v_{\kappa|U_1}$ from U_1 to Ω, the extension being denoted by f_κ, such that, for each $m \in I\!N$, we have

$$f_\kappa = v_\kappa \text{ in } U_1,\ f_\kappa = 1 \text{ in } \Omega \setminus U_2,\ \|f - 1\|_{C^l(K)} = o\Big(\frac{1}{\kappa^m}\Big), \text{ as } \kappa \to \infty,\ \forall\, l \in I\!N,\ \forall\, K \text{ compact in } \Omega. \tag{11.69}$$

Similarly, we may find, in $\mathbb{C} \setminus \omega_0$, a map g_κ such that, for each $m \in I\!N$, we have

$$g_\kappa = v_\kappa \text{ in } V_1,\ f_\kappa = 1 \text{ in } \mathbb{C} \setminus V_2,\ \|f - 1\|_{C^l(K)} = o\Big(\frac{1}{\kappa^m}\Big), \text{ as } \kappa \to \infty,\ \forall\, l \in I\!N,\ \forall\, K \text{ compact in } \mathbb{C} \setminus \omega_0. \tag{11.70}$$

Note that, by construction, we have

$$u_\kappa = \beta_\kappa f_\kappa g_\kappa \quad \text{ in } U_1 \cup V_1. \tag{11.71}$$

By Lemma 11.1, we also have

$$\|u_\kappa - \beta_\kappa f_\kappa g_\kappa\|_{H^1(A)} = o\Big(\frac{1}{\kappa^m}\Big), \text{ as } \kappa \to \infty,\ \forall\, m \in I\!N. \tag{11.72}$$

Moreover, by Lemma 11.6, we obtain

$$\int_\Omega \Big(\frac{1}{2}|\nabla f_\kappa|^2 + \frac{\kappa^2}{4}(1 - |f_\kappa|^2)^2\Big) = \pi + o\Big(\frac{1}{\kappa^m}\Big), \quad \text{ as } \kappa \to \infty \tag{11.73}$$

and

$$\int_{\mathbb{C} \setminus \omega_0} \Big(\frac{1}{2}|\nabla g_\kappa|^2 + \frac{\kappa^2}{4}(1 - |g_\kappa|^2)^2\Big) = \pi + o\Big(\frac{1}{\kappa^m}\Big), \quad \text{ as } \kappa \to \infty. \tag{11.74}$$

Note also that, by construction, the zeroes of u_κ which are close to $\partial\Omega$ coincide, for large κ, with the zeroes of f_κ, and the zeroes of u_κ close to $\partial\omega_0$ with the ones of g_κ. We next note that, by the proof of lemma 4.4, for a fixed κ, the zeroes of a critical point u_κ of E_κ in $\mathcal{J}$ can not tend to ∂A, applies also to the case of the quasi-minimizers. Indeed, the argument in the proof of Lemma 4.4 requires only that $|u_\kappa| = 1$ on ∂A and that u_κ satisfies the Ginzburg-Landau equation, which is the case for quasi-minimizers. Therefore, for large κ, there are two (possibly not unique) points in A, $\zeta = \zeta_\kappa$, $\xi = \xi_\kappa$, such that

$$u_\kappa(\zeta) = f_\kappa(\zeta) = 0 \quad \text{ and } \quad u_\kappa(z) = 0 \Longrightarrow\ \mathrm{dist}(z, \partial\Omega) \geq \mathrm{dist}(\zeta, \partial\Omega) \tag{11.75}$$

and respectively

$$u_\kappa(\xi) = g_\kappa(\xi) = 0 \quad \text{and} \quad u_\kappa(z) = 0 \Longrightarrow \operatorname{dist}(z, \partial\omega_0) \geq \operatorname{dist}(\xi, \partial\omega_0). \tag{11.76}$$

Moreover, by Lemma 11.3, these zeroes are very close to $\partial\Omega$ or $\partial\omega_0$, more specifically

$$\operatorname{dist}(\zeta, \partial\Omega) = o\Big(\frac{1}{\kappa^m}\Big), \quad \operatorname{dist}(\xi, \partial\omega_0) = o\Big(\frac{1}{\kappa^m}\Big), \quad \text{as } \kappa \to \infty, \forall\, m \in I\!N. \tag{11.77}$$

The following result is the main step in establishing (P3).

Lemma 11.7. *Let $\kappa_l \to \infty$. Then there is some $\alpha \in S^1$ such that, up to some subsequence, $f_{\kappa_l} \circ (F^{\zeta_{\kappa_l}})^{-1} \to \alpha\, id$ strongly in $H^1(I\!D)$.*

Proof : We split the proof into several steps.
Step 1. Existence of a holomorphic weak H^1 limit w for $f_{\kappa_l} \circ (F^{\zeta_{\kappa_l}})^{-1}$
Set $a = a_\kappa = f \circ (F^\zeta)^{-1} = f_\kappa \circ (F^{\zeta_\kappa})^{-1} : \overline{I\!D} \to \mathbb{C}$. Since F^ζ is a conformal representation, we have

$$\int_{I\!D} |\nabla a|^2 = \int_\Omega |\nabla f|^2 = 2\pi + o\Big(\frac{1}{\kappa^m}\Big), \quad \text{as } \kappa \to \infty, \forall\, m \in I\!N, \tag{11.78}$$

by (11.73).
Using again the fact that F^ζ is a conformal representation (and thus an orientation preserving diffeomorphism), we also have

$$\int_{I\!D} (|\nabla a|^2 - 2\text{Jac } a) = \int_\Omega (|\nabla f|^2 - 2\text{Jac } f) = o\Big(\frac{1}{\kappa^m}\Big), \quad \text{as } \kappa \to \infty, \forall\, m \in I\!N, \tag{11.79}$$

by (11.69) and Lemma 11.5.
Also note that

$$|a| = 1 \quad \text{on } S^1. \tag{11.80}$$

From (11.78) and (11.80) it follows that, up to subsequences, $a \rightharpoonup w$ weakly in $H^1(I\!D)$ for some map w such that $|w| = 1$ on S^1. Moreover, since the map

$$H^1(A) \ni u \mapsto \int_A (|\nabla u|^2 - 2 \text{ Jac } u)$$

is convex and continuous, (11.79) together with the fact that $a \rightharpoonup w$ weakly in $H^1(I\!D)$ imply that

$$\int_{I\!D} (|\nabla w|^2 - 2\text{Jac } w) \leq 0. \tag{11.81}$$

By (11.81) and the identity

$$|\nabla w|^2 - 2\text{Jac } w = \Big(\frac{\partial \operatorname{Re} w}{\partial x} - \frac{\partial \operatorname{Im} w}{\partial y}\Big)^2 + \Big(\frac{\partial \operatorname{Re} w}{\partial y} + \frac{\partial \operatorname{Im} w}{\partial x}\Big)^2, \tag{11.82}$$

it follows that, in the distribution sense, we have

$$\frac{\partial \operatorname{Re} w}{\partial x} = \frac{\partial \operatorname{Im} w}{\partial y} \quad \text{and} \quad \frac{\partial \operatorname{Re} w}{\partial y} = -\frac{\partial \operatorname{Im} w}{\partial x}, \tag{11.83}$$

so that w is holomorphic in $I\!\!D$. Moreover, it follows from (11.78) that

$$\int_{I\!\!D} |\nabla w|^2 \leq 2\pi. \tag{11.84}$$

Set $b = \text{tr}_{S^1} w \in H^{1/2}(S^1; S^1)$. Since w is holomorphic, the Fourier expansion of b is of the form $\sum_{l=0}^{\infty} c_l e^{\imath l\theta}$ and, by the degree formula (1.6), its degree is

$$\deg b = \sum_{l=1}^{\infty} l|c_l|^2. \tag{11.85}$$

On the other hand, since w is holomorphic, it coincides with the harmonic extension of b, and thus, after a simple computation, we find

$$\int_{I\!\!D} |\nabla w|^2 = 2\pi \sum_{l=1}^{\infty} l|c_l|^2. \tag{11.86}$$

If we compare (11.84), (11.85) and (11.86), we find that either $\deg b = 0$, and then b (and thus w) has to be a constant of modulus 1, or $\deg b = 1$, and then

$$\int_{I\!\!D} |\nabla w|^2 = 2\pi. \tag{11.87}$$

Step 2. w is not constant
We start by excluding the first possibility, i.e., we prove that w is not a constant of modulus 1. Intuitively speaking, this comes from the fact that $a(0) = 0$.
Fix some $0 < \varepsilon < 1$. We will find some $\delta > 0$ such that, for all sufficiently large κ, $|a(z)| \leq \varepsilon$ if $|z| \leq \delta$. Set $d = d_\kappa = \text{dist}(\zeta, \partial\Omega) = \text{dist}(\zeta_\kappa, \partial\Omega)$. It is clear from the definition of F^ζ (and explained in Appendix X) that there are some constants $C(\delta) > 0$, for $0 < \delta < 1$, independent of large κ, such that

$$|(F^\zeta)^{-1}(z) - (F^\zeta)^{-1}(0)| = |(F^\zeta)^{-1}(z) - \zeta| \leq C(\delta) d|z| \quad \text{if } |z| \leq \delta. \tag{11.88}$$

On the other hand, recall the estimate (11.29), which with our notations becomes, for large κ,

$$|\nabla f(z)| = |\nabla u_\kappa(z)| \le \frac{D}{d} \quad \text{ if } |z-\zeta| \le \frac{d}{2}, \tag{11.89}$$

for some constant D independent of κ. By combining (11.88) and (11.89), we see that, for each $\varepsilon > 0$, there is some $\delta = \delta(\varepsilon) > 0$ such that $|a(z)| \le \varepsilon$ if $|z| \le \delta$.
As a consequence, $|w(z)| \le \varepsilon$ if $|z| \le \delta$. Therefore, w is not a constant, as claimed. Moreover, since w is holomorphic, and thus smooth in $I\!\!D$, we may also infer the fact that

$$w(0) = 0. \tag{11.90}$$

Step 3. Conclusion
We know that w is a holomorphic map in $I\!\!D$ such that $w(0) = 0$ and $|w| = 1$ on S^1. Therefore, there is some $\alpha \in S^1$ such that $w = \alpha\, id$, by the Schwartz lemma (see, e.g., [1]). On the other hand, by (11.78) and (11.87) we find that $\int\limits_{I\!\!D} |\nabla a|^2 \to \int\limits_{I\!\!D} |\nabla w|^2$. Therefore, up to subsequences, a **strongly** converges in $H^1(I\!\!D)$ to $\alpha\, id$, as claimed.

Similarly, we have

Lemma 11.8. *Let $\kappa_l \to \infty$. Then there is some $\beta \in S^1$ such that, up to some subsequence, $g_{\kappa_l} \circ \overline{(G^{\xi_{\kappa_l}})}^{-1} \to \beta\ id$ strongly in $H^1(I\!\!D)$.*

We may now prove the precise statement concerning property (P3), which is a trivial consequence of Lemmas 11.7 and 11.8 :

Lemma 11.9. *Let (u_κ) be a family of quasi-minimizers. Let U, V be two smooth open sets such that (11.62) holds. Then there are constants $\alpha_\kappa \in S^1$, $\gamma_\kappa \in S^1$ such that*

$$\int\limits_U |\nabla(u_\kappa - \alpha_\kappa F^\zeta)|^2 \to 0 \quad \text{ as } \kappa \to \infty \tag{11.91}$$

and respectively

$$\int\limits_V |\nabla(u_\kappa - \gamma_\kappa \overline{G^\xi})|^2 \to 0 \quad \text{ as } \kappa \to \infty. \tag{11.92}$$

Proof : By Lemma 11.7, we have

$$\lim_{\kappa\to\infty} \inf_{\alpha \in S^1} \int\limits_{I\!\!D} |\nabla(a_\kappa - \alpha\, id)|^2 = 0. \tag{11.93}$$

Thus, we may find a family $(\delta_\kappa) \subset S^1$ such that

$$\lim_{\kappa\to\infty} \int_{I\!D} |\nabla(a_\kappa - \delta_\kappa\, id)|^2 = 0. \tag{11.94}$$

In turn, this implies, since F^ζ is a conformal representation, that

$$\lim_{\kappa\to\infty} \int_{\Omega} |\nabla(f_\kappa - \delta_\kappa\, F^{\zeta_\kappa})|^2 = 0. \tag{11.95}$$

By (11.70) and the fact that, by construction, $\nabla F^\zeta \to 0$ on compact subsets of Ω, this immediately implies (11.91) for $\alpha_\kappa = \beta_\kappa \delta_\kappa$, where the β_κ are given by Lemma 11.3. The proof of (11.92) is similar.

We now start the proof of (P1). The first step consists in proving that the energy of u_κ is essentially concentrated in a ball of radius of order $\mathrm{dist}(\zeta, \partial\Omega)$ around ζ and in a ball of radius of order $\mathrm{dist}(\xi, \partial\omega_0)$ around ξ.

Lemma 11.10. *Let $\delta > 0$. Then there are constants $R_\delta > 0$ and $r_\delta > 0$ such that, for sufficiently large κ, we have*

$$\frac{1}{2}\int_{U(\zeta)} |\nabla u_\kappa|^2 \geq \pi - \frac{\delta}{3}, \tag{11.96}$$

$$\frac{1}{2}\int_{V(\xi)} |\nabla u_\kappa|^2 \geq \pi - \frac{\delta}{3} \tag{11.97}$$

and

$$\frac{1}{2}\int_{A\setminus(U(\zeta)\cup V(\xi))} |\nabla u_\kappa|^2 \leq \delta. \tag{11.98}$$

Here,

$$U(\zeta) = \{z \in A\,;\, |z-\zeta| \leq R_\delta \mathrm{dist}(\zeta, \partial\Omega)\,,\, \mathrm{dist}(z, \partial\Omega) \geq r_\delta \mathrm{dist}(\zeta, \partial\Omega)\} \tag{11.99}$$

and

$$V(\xi) = \{z \in A\,;\, |z-\xi| \leq R_\delta \mathrm{dist}(\xi, \partial\omega_0)\,,\, \mathrm{dist}(z, \partial\omega_0) \geq r_\delta \mathrm{dist}(\xi, \partial\omega_0)\}. \tag{11.100}$$

Proof : Estimate (11.98) follows from (11.96) and (11.97) using the upper bound (11.6). We will prove only (11.96), the proof of (11.97) being similar. Let $\delta > 0$. Fix some radius $0 < \rho < 1$ such that

$$\int_{\{z\,;\,|z|<\rho\}} |\nabla id|^2 > \pi - \frac{\delta}{3}. \tag{11.101}$$

By Lemma 11.7, for sufficiently large κ we have

$$\int\limits_{\{z\,;\,|z|<\rho\}} |\nabla a|^2 > \pi - \frac{\delta}{3}. \tag{11.102}$$

Since F^ζ is a conformal representation, we find that

$$\int\limits_{(F^\zeta)^{-1}(\{z\,;\,|z|<\rho\})} |\nabla u_\kappa|^2 = \int\limits_{(F^\zeta)^{-1}(\{z\,;\,|z|<\rho\})} |\nabla f|^2 = \int\limits_{\{z\,;\,|z|<\rho\}} |\nabla a|^2 > \pi - \frac{\delta}{3}. \tag{11.103}$$

The conclusion of Lemma 11.10 follows then trivially from the fact that, by the definition of F^ζ, there are clearly some constants $R_\delta > 0$ and $r_\delta > 0$ such that, for sufficiently large κ, we have (see also Appendix X):

$$(F^\zeta)^{-1}(\{z\,;\,|z|<\rho\}) \subset \{z \in A\,;\, |z-\zeta| \leq R_\delta \mathrm{dist}(\zeta, \partial\Omega)\,,\, \mathrm{dist}(z, \partial\Omega) \geq r_\delta \mathrm{dist}(\zeta, \partial\Omega)\}. \tag{11.104}$$

We next prove that, if u_κ has, near $\partial\Omega$, a zero different from ζ, then that zero is "far away" from ζ in a suitable scale.

Lemma 11.11. *Let $R > 0$. Then, for sufficiently large κ, the only zero of u_κ in the set $\{z \in A\,;\, \mathrm{dist}(z,\zeta) \leq R\,\mathrm{dist}(\zeta, \partial\Omega)\}$ is ζ.*

Proof : By our choice of ζ as the zero of u_κ closest to $\partial\Omega$, we have $u_\kappa(z) \neq 0$ if $\mathrm{dist}(z, \partial\Omega) \leq \mathrm{dist}(\zeta, \partial\Omega)$. Therefore, it suffices to prove that ζ is the only zero of u_κ in the set

$$B = B_\kappa = \{z \in A\,;\, |z-\zeta| \leq R\,\mathrm{dist}(\zeta, \partial\Omega),\, \mathrm{dist}(z, \partial\Omega) \geq \mathrm{dist}(\zeta, \partial\Omega)\}. \tag{11.105}$$

By the form of F^ζ (see also Appendix X), there is some compact smooth convex $K \subset I\!\!D$, independent of κ, such that $B \subset (F^\zeta)^{-1}(K)$. On the other hand, using again the form of F^ζ, there are constants $\rho > 0$, $r > 0$ such that

$$B \subset (F^\zeta)^{-1}(K) \subset C = C_\kappa = \{z \in A\,;\, |z-\zeta| \leq \rho\,\mathrm{dist}(\zeta, \partial\Omega),\, \mathrm{dist}(z, \partial\Omega) \geq r\,\mathrm{dist}(\zeta, \partial\Omega)\}. \tag{11.106}$$

The conclusion of Lemma 11.11 is an immediate consequence of the following

Claim. For sufficiently large κ, the restriction of u_κ to B is a C^1 diffeomorphism into its image.

Proof of the Claim : Split $u_\kappa = v_1 + v_2$, where

$$\begin{cases} -\Delta v_1 &= 0 \quad \text{in } A \\ \quad\; v_1 &= u_\kappa \quad \text{on } \partial A \end{cases} \tag{11.107}$$

and

$$\begin{cases} -\Delta v_2 & = h = h_\kappa = \kappa^2 u_\kappa (1 - |u_\kappa|^2) \quad \text{in } A \\ \quad v_2 & = 0 \qquad\qquad\qquad\qquad\qquad\quad\;\, \text{on } \partial A \end{cases} . \tag{11.108}$$

Since v_1 is harmonic and $|v_1| = 1$ on ∂A, we find, by standard estimates for the Green function, that

$$|\nabla v_1(z)| \leq \frac{C_1}{\mathrm{dist}(z, \partial\Omega)} \quad \text{and } |D^2 v_1(z)| \leq \frac{C_1}{(\mathrm{dist}(z, \partial\Omega))^2} \quad \text{in } A, \tag{11.109}$$

for some C_1 independent of large κ. Taking the definition of the set C into account, this implies that

$$|\nabla v_1(z)| \leq \frac{C_2}{\mathrm{dist}(\zeta, \partial\Omega)} \quad \text{and } |D^2 v_1(z)| \leq \frac{C_2}{(\mathrm{dist}(\zeta, \partial\Omega))^2} \quad \text{in } C. \tag{11.110}$$

By interpolation, this yields, with some constant C_3 independent of large κ, the estimate

$$|\nabla v_1(z) - \nabla v_1(z')| \leq \frac{C_3 |z - z'|^b}{(\mathrm{dist}(\zeta, \partial\Omega))^{1+b}} \quad \text{in } C, \forall\, 0 < b < 1. \tag{11.111}$$

On the other hand, we find from Lemma 11.1 and the inequality $|u_\kappa| \leq 1$, valid for quasi-minimizers, that $\|h\|_{L^p(A)} \to 0$ as $\kappa \to \infty$, and thus $\|v_2\|_{W^{2,p}(A)} \to 0$ as $\kappa \to \infty$, for $1 < p < \infty$. By the Sobolev embedings, this implies that

$$\|v_2\|_{C^{1,b}(\overline{A})} \to 0 \quad \text{as } \kappa \to \infty, \forall\, 0 < b < 1. \tag{11.112}$$

By combining (11.110), (11.111) and (11.112), we find that

$$|\nabla u_\kappa(z)| \leq \frac{C_4}{\mathrm{dist}(\zeta, \partial\Omega)} \quad \text{and } |\nabla u_\kappa(z) - \nabla u_\kappa(z')| \leq \frac{C_4 |z - z'|^b}{(\mathrm{dist}(\zeta, \partial\Omega))^{1+b}} \quad \text{in } C, \forall\, 0 < b < 1, \tag{11.113}$$

where C_4 depends only on b.

We next transport the above estimates to $I\!\!D$ with the help of F^ζ. Note that, by construction, we have

$$|\nabla (F^\zeta)^{-1}(z)| \leq C_5 \mathrm{dist}(\zeta, \partial\Omega) \quad \text{and} \quad |D^2 (F^\zeta)^{-1}(z)| \leq C_5 \mathrm{dist}(\zeta, \partial\Omega) \quad \text{in } K. \tag{11.114}$$

By (11.113) and (11.114), we find immediately, with $a = a_\kappa = u_\kappa \circ (F^\zeta)^{-1}$, that

$$|\nabla a(z) - \nabla a(z')| \leq C_6 (|z - z'| + |z - z'|^b) \leq C_7 |z - z'|^b \quad \text{in } K, \forall\, 0 < b < 1, \tag{11.115}$$

where C_7 depends only on b.

Therefore, the family (a_κ) is relatively compact in $C^{1,b}(K)$, for $0 < b < 1$. On the other hand, recall that, by Lemma 11.7, up to subsequences, a_κ converges strongly in $H^1(A)$ to $\alpha\, id$, for some $\alpha \in S^1$. Therefore, for sufficiently large κ, a_κ is a C^1 diffeomorphism in K. Since F^ζ is a diffeomorphism, we find that $a \circ F^\zeta$ is a diffeomorphism in $(F^\zeta)^{-1}(K)$; in particular, u_κ is a C^1 diffeomorphism in B. This completes the proof of Lemma 11.11.

Remark 11.1. With a little more work, the above proof yields the following estimate :

$$|u_\kappa(z)| \geq \frac{C_8|z-\zeta|}{\mathrm{dist}(\zeta,\partial\Omega)} \quad \text{in } B, \tag{11.116}$$

for some constant independent of large κ.

We may now establish property (P1), which we state as

Proposition 11.1. *Assume that $I_0 > 2\pi$. Then each quasi-minimizer u_κ has exactly two zeroes, provided κ is sufficiently large.*

Proof : It suffices to reason near $\partial\Omega$, the proof being similar near $\partial\omega_0$. Argue by contradiction and assume that, along some sequences, there is some $\lambda = \lambda_\kappa \neq \zeta$ such that $\mathrm{dist}(\lambda,\partial\Omega) \to 0$ as $\kappa \to \infty$ and $u_\kappa(\lambda) = 0$. Let $0 < \delta < \pi$. With the notations of Lemma 11.10, there are some $R_\delta > 0$ and $r_\delta > 0$ such that, for sufficiently large κ, we have

$$\frac{1}{2}\int_{U(\zeta)} |\nabla u_\kappa|^2 \geq \pi - \frac{\delta}{3} > \frac{2\pi}{3}. \tag{11.117}$$

Since the proof of Lemma 11.10 does not use the fact that ζ is the zero of u_κ closest to $\partial\Omega$, we also have

$$\frac{1}{2}\int_{U(\lambda)} |\nabla u_\kappa|^2 \geq \pi - \frac{\delta}{3} > \frac{2\pi}{3}. \tag{11.118}$$

From (11.117) and (11.118) it follows that, for large κ, we must have $U(\zeta)\cap U(\lambda) \neq \emptyset$, for otherwise we would contradict the conclusion of Lemma 11.6. Let $z \in U(\zeta)\cap U(\lambda)$. It follows from the definition of $U(w)$ that

$$|\zeta-\lambda| \leq |\zeta - z| + |z-\lambda| \leq R_\delta \mathrm{dist}(\zeta,\partial\Omega) + R_\delta \mathrm{dist}(\lambda,\partial\Omega), \tag{11.119}$$

so that

$$|\zeta-\lambda| \leq R_\delta \mathrm{dist}(\zeta,\partial\Omega) + \frac{R_\delta}{r_\delta}\mathrm{dist}(z,\partial\Omega) \leq R_\delta \mathrm{dist}(\zeta,\partial\Omega) + \frac{R_\delta^2}{r_\delta}\mathrm{dist}(\zeta,\partial\Omega) \equiv R\mathrm{dist}(\zeta,\partial\Omega). \tag{11.120}$$

This contradicts the conclusion of Lemma 11.11 for large κ. The proof of Proposition 11.1 is complete.

Remark 11.2. With more work, one may prove that $|u_\kappa|$ is close to 1 if we are far away from from ζ and ξ in the respective scales $\mathrm{dist}(\zeta,\partial\Omega)$ and $\mathrm{dist}(\xi,\partial\omega_0)$, that is : for each $0 < \delta < 1$, there are constants R_δ and r_δ such that, for large κ, we have $|u_\kappa| \geq \delta$ in the set $A \setminus (U(\zeta)\cup V(\xi))$.

We end this section by establishing the property (P4).

Lemma 11.12. *For sufficiently large κ, ζ_κ is a zero of degree 1 of u_κ, while ξ_κ is a zero of degree -1 of u_κ.*

Proof : We reason for ζ, the argument being similar for ξ. Fix some $0 < r <$dist$(\zeta, \partial\Omega)$. We have to prove that $\deg(u_\kappa, \mathcal{C}) = 1$, where $\mathcal{C} = \{z \,|z - \zeta| = r\}$ and the orientation on $\mathcal{C}$ is the direct one. Fix some sufficiently small $\delta > 0$ (independent of κ). Let

$$U = \{z \in A\,;\, \text{dist}(z, \partial A) < \delta,\ |z - \zeta| > r\}. \tag{11.121}$$

As in the proof of Lemma 7.2, for sufficiently large κ there is some $D > 0$ such that $D < |u_\kappa| \le 1$ in U. Arguing again as in the proof of Lemma 7.2, the map $v = u_\kappa/|u_\kappa|$ belongs to $H^1(U; S^1)$, and if we orient the components of ∂U with the orientations inherited from U, we have $\deg(v, \Gamma) = 0$, $\deg(v, \partial\Omega) = 1$ and $\deg(v, \mathcal{C}) = -\deg(u_\kappa, \mathcal{C})$. By Lemma 2.2 applied to U, we find that $\deg(v, \mathcal{C}) = -1$, so that $\deg(u_\kappa, \mathcal{C}) = 1$.

12 Existence of stable critical points in the supercritical case

From what we know by now, if A is supercritical, then : either (i) there is some finite κ_1 such that the minimizers of (1.1)-(1.3) do not exist for $\kappa > \kappa_1$ or (ii) for large κ, the minimizers have two zeroes. In this section, we prove that, for large values of κ, there are locally minimizing critical points of E_κ that do not vanish. In particular, these critical points are not minimizers. The construction we present below is also valid when A is sub critical or critical, but it is easy to see that in these cases it actually yields minimizers. The same idea could be used to obtain stable solution of the Neumann problem, in the spirit of [28] and [29]. However, the method in [28] seems to be very much related to radial symmetry, while our works in general domains.

We will need the following variant of Lemma 8.2 :

Lemma 12.1. *Let u be a minimizer of (1.14)-(1.15). Let $(u_n) \subset \mathcal{K}$ be such that $u_n \to u$ strongly in $H^1(A)$. Set $g_n =$tr$_{\partial A} u_n$, $g =$tr$_{\partial A} u$. Let $\kappa_n \to \infty$ and, for each n, let u^n be a minimizer of E_{κ_n} in the class*

$$\{u \in H^1(A)\,;\, \text{tr}_{\partial A} u = g_n\}.$$

Then $|u^n| \to 1$ uniformly in $\overline{A}$ as $n \to \infty$.

Proof : All we need in order to be able to repeat the proof of Lemma 8.2 is the estimate

$$\lim_{n\to\infty} \kappa_n^2 \int_A (1 - |u^n|^2)^2 = 0. \tag{12.1}$$

Since

$$E_{\kappa_n}(u^n) \le E_{\kappa_n}(u_n) = \frac{1}{2}\int_A |\nabla u_n|^2 \to \frac{1}{2}\int_A |\nabla u|^2 = I_0, \tag{12.2}$$

we find that, up to subsequences, $u^n \rightharpoonup v$ weakly in $H^1(A)$ to some $v \in H^1(A; S^1)$ such that

$$\frac{1}{2}\int_A |\nabla v|^2 \leq \frac{1}{2}\int_A |\nabla u|^2. \tag{12.3}$$

Since we also have $\mathrm{tr}_{\partial A} v = \mathrm{tr}_{\partial A} u$, it follows that $v \in \mathcal{K}$; therefore, $v = u$, by Lemma 2.4. Going back to (12.2), we find that $u^n \to u$ strongly in $H^1(A)$ and (12.1) follows by using again (12.2).

We may now start our construction of stable critical points. Set

$$\mathcal{J}' = \{u \in \mathcal{J}\, ;\, 1/2 \leq |u| \leq 2 \text{ in } \overline{A}\} \tag{12.4}$$

and let

$$n_\kappa = \mathrm{Inf}\{E_\kappa(u)\, ;\, u \in \mathcal{J}'\}. \tag{12.5}$$

Lemma 12.2. *n_κ is attained.*

Proof : Let $u \in \mathcal{J}'$. Set $\rho = |u|$ and $v = u/|u|$. Clearly, $\rho \in H^1(A)$ and $\mathrm{tr}_{\partial A}\rho = 1$. Moreover, since the map $f(z) = z/|z|$ has a bounded differential in $\{z\, ;\, |z| \geq 1/2\}$, it follows that we also have $v \in H^1(A)$; more specifically, we have $v \in \mathcal{K}$. On the other hand, we have

$$\nabla u = v\nabla u + \rho\nabla v, \quad v\nabla\overline{v} + \overline{v}\nabla v = 0 \quad \text{ and } |\nabla u|^2 = |\nabla\rho|^2 + \rho^2|\nabla v|^2,\ \forall\, u \in \mathcal{J}'. \tag{12.6}$$

Indeed, the second equality in (12.6) follows by differentiating the identity $v\overline{v} \equiv 1$; here, we use the fact that

$$\nabla(fg) = f\nabla g + g\nabla f \quad \text{ if } f, g \in H^1 \cap L^\infty\, ; \tag{12.7}$$

see, e.g., [14]. The third equality is a straightforward consequence of the first two, so it remains to establish the first equality in (12.6). This identity is clear since, for $u \in \mathcal{J}'$, ρ is bounded, and thus we may use (12.7).

Let now (u^n) be a minimizing sequence for n_κ. We may write $u^n = \rho^n v^n$, with $v^n \in \mathcal{K}$, $\rho^n \in H^1(A)$, $1/2 \leq \rho^n \leq 2$, $\mathrm{tr}_{\partial A}\rho^n = 1$. By (12.6), we have

$$\frac{1}{2}\int_A |\nabla\rho^n|^2 + \frac{1}{8}\int_A |\nabla v^n|^2 \leq \int_A E_\kappa(u^n) = \frac{1}{2}\int_A |\nabla\rho|^2 + \frac{1}{2}\int_A (\rho^n)^2|\nabla v^n|^2 + \frac{\kappa^2}{4}\int_A (1-(\rho^n)^2)^2 \leq I_0 + o(1)\, ; \tag{12.8}$$

the last inequality comes from the fact that any minimizer u of I_0 belongs to $\mathcal{J}'$, so that $n_\kappa \leq I_0$. It follows that the sequences (ρ^n) and (v^n) are bounded in $H^1(A)$ (for ρ^n, we also use the fact that it has trace 1). Up to some sequence, we may find some $\rho, v \in H^1(A)$ such that $\rho^n \rightharpoonup \rho$, $v^n \rightharpoonup v$ weakly in $H^1(A)$. On the one hand, we clearly have $1/2 \leq \rho \leq 1$. On the other hand, recall that, by Lemma 2.3 b), the class $\mathcal{K}$ is closed with respect to weak H^1 convergence, and therefore $v \in \mathcal{K}$. Set $u = \rho v$. It is then straightforward that $u \in \mathcal{J}'$ and that u is a minimizer of (12.5), which completes the proof of the lemma.

¿From now on, we will denote by u_κ a minimizer of (12.5) and we set $g_\kappa = \mathrm{tr}_{\partial A} u_\kappa$. With obvious notations, we may write $u_\kappa = \rho_\kappa v_\kappa$, and then $g_\kappa = \mathrm{tr}_{\partial A} v_\kappa$.

Lemma 12.3. *Up to subsequences, we have $\rho_\kappa \to 1$, $v_\kappa \to u$ and $u_\kappa \to u$ strongly in $H^1(A)$, where u is a minimizer of (1.14)-(1.15).*

Proof : As in the proof of Lemma 12.2, starting from

$$E_\kappa(u_\kappa) \leq I_0, \tag{12.9}$$

we may derive that, up to subsequences, $u_\kappa \to u$ strongly in $H^1(A)$, for some minimizer u of I_0. Therefore, $\rho_\kappa = |u_\kappa| \to |u| = 1$ strongly in $H^1(A)$. Finally, the two previous convergences combined with the fact that $v_\kappa = u_\kappa \dfrac{1}{\rho_\kappa}$ yield $v_\kappa \to u$ strongly in $H^1(A)$; here, we use the fact that $H^1 \cap L^\infty$ is an algebra.

Lemma 12.4. *There is some $\kappa_0 > 0$ such that, for $\kappa > \kappa_0$, u_κ attains*

$$\mathrm{Min}\{E_\kappa(w)\,;\, \mathrm{tr}_{\partial A} w = g_\kappa\}. \tag{12.10}$$

Proof : Let u^κ attain the above minimum. By combining Lemma 12.1 and Lemma 12.3, we find that $|u^\kappa| \to 1$ uniformly in $\overline{A}$. Therefore, $u^\kappa \in \mathcal{J}'$ for large κ and thus $E_\kappa(u^\kappa) \geq E_\kappa(u_\kappa)$. The opposite inequality $E_\kappa(u^\kappa) \leq E_\kappa(u_\kappa)$ being clear from the definition of u^κ, we find that $E_\kappa(u^\kappa) = E_\kappa(u_\kappa)$, and thus u_κ attains the minimum in the statement of the Lemma.

Lemma 12.5. *For large κ, u_κ is a critical point of E_κ in $\mathcal{J}$.*

Proof : By the preceding Lemma, for large κ, u_κ satisfies the Ginzburg-Landau equation, since it is a minimizer of E_κ with respect to its own boundary trace. On the other hand, if $\psi \in C^\infty(\overline{A}; I\!R)$, then $u_\kappa e^{\imath t\psi} \in J'$, and thus $E_\kappa(u_\kappa e^{\imath t\psi}) \geq E_\kappa(u_\kappa)$, $t \in I\!R$. This inequality together with the fact that u_κ satisfies the Ginzburg-Landau equation implies immediately that, in the weak sense, we have $u_\kappa \wedge \dfrac{\partial u_\kappa}{\partial \nu} = 0$ on ∂A, that is, u_κ is a critical point of E_κ in $\mathcal{J}$.

We are now ready to prove that the above u_κ's are local minimizers of E_κ in $\mathcal{J}$.

Proposition 12.1. *There are constants $\kappa' > 0$ and $\delta > 0$ such that :*
a) if $\kappa > \kappa'$ and if $v \in \mathcal{J}$ is such that $\|u_\kappa - v\|_{H^1(A)} < \delta$, then $E_\kappa(u_\kappa) \leq E_\kappa(v)$;
b) if $\kappa > \kappa'$, then v_κ is a minimizer of (12.5) if and only if there is some $\alpha \in S^1$ such that $v_\kappa = \alpha u_\kappa$.

Proof : For b), we argue as as in the proof of Lemma 10.2. As explained there, since, up to subsequences, $u_\kappa \to u$ strongly in $H^1(A)$, we find subsequently, as in Section 9 that $u_\kappa \to u$ in $C^{1,b}(\overline{A})$, $0 < b < 1$. This is what is needed to derive inequality (10.11), i.e, $E_\kappa(v_\kappa) \geq E_\kappa(u_\kappa)$, with equality if and only if $v_\kappa = \alpha u_\kappa$ for some $\alpha \in S^1$.

As for the proof of a), we argue by contradiction. Assume that there is a sequence $\kappa_n \to \infty$ and that there are maps $v_{\kappa_n} \in \mathcal{J}$ such that $\|u_{\kappa_n} - v_{\kappa_n}\|_{H^1(A)} < 1/n$ and $E_{\kappa_n}(v_{\kappa_n}) < E_{\kappa_n}(u_{\kappa_n})$. Passing to a subsequence, we may assume that $u_{\kappa_n} \to u$ strongly in $H^1(A)$, where u is a minimizer of (1.14)-(1.15). Let w_{κ_n} attain the minimum of E_κ among all the functions that agree with v_{κ_n} on ∂A ; thus $E_{\kappa_n}(w_{\kappa_n}) \leq E_{\kappa_n}(v_{\kappa_n}) < E_{\kappa_n}(u_{\kappa_n})$. Starting from

$$E_{\kappa_n}(w_{\kappa_n}) < E_{\kappa_n}(u_{\kappa_n}) \leq I_0, \tag{12.11}$$

we find that, up to subsequences, $w_{\kappa_n} \rightharpoonup v$ weakly in $H^1(A)$ to some $v \in H^1(A; S^1)$. Moreover, by taking traces, we have $\text{tr}_{\partial A} v = \text{tr}_{\partial A} u$, so that $v \in \mathcal{K}$. As in the proof of Lemma 12.1, this implies that $v = u$ and that $w_{\kappa_n} \to u$ strongly in $H^1(A)$. By the proof of Lemma 8.2, this implies that $|w_{\kappa_n}| \to 1$ uniformly in $\overline{A}$ as $n \to \infty$. Therefore, for large n we have $w_{\kappa_n} \in \mathcal{J}'$, and thus $E_{\kappa_n}(w_{\kappa_n}) \geq E_{\kappa_n}(u_{\kappa_n})$, which is the desired contradiction.

As explained at the end of Section 10, the above result implies the following

Corollary 12.1. *Assume that A is $\mathcal{O}$-symmetric in the sense of Definition 2.3. Then, for large κ, there is a local minimizer of E_κ in $\mathcal{J}$ such that $u_\kappa \circ \mathcal{O} = \mathcal{O} \circ u_\kappa$.*

Assume now that A is a circular annulus, $A = \{z\,;\, \rho < |z| < R\}$. Arguing as at the end of the Section 10, we obtain the existence of a local minimizer E_κ in $\mathcal{J}$ of the form $u_\kappa(z) = f(|z|)\dfrac{z}{|z|}$; moreover, we may assume that $f(\rho) = f(R) = 1$. In particular, this u_κ must satisfy the Ginzburg-Landau equation. However, it is well known that there is exactly one f such that $f(\rho) = f(R) = 1$ and such that u_κ satisfies the Ginzburg-Landau equation ; see, e.g., [27]. We are thus led to the following

Corollary 12.2. *Assume that $A = \{z\,;\, \rho < |z| < R\}$. Then, for large κ, the only solution of the Ginzburg-Landau equation of the form $u_\kappa(z) = f(|z|)\dfrac{z}{|z|}$, where $f(\rho) = f(R) = 1$, is a local minimizer of E_κ in $\mathcal{J}$.*

Appendix A. Degree of $H^{1/2}$ maps and capacity

We prove below some results stated in Section 2.

Proof of Lemma 2.1 : When $u \in C^\infty(\overline{A}; \mathbb{C})$, the above equality is clear by integration by parts. The case of a general $u \in H^1(A; \mathbb{C})$ follows by considering a sequence $(u_n) \subset C^\infty(\overline{A}; \mathbb{C})$ such that $u_n \to u$ strongly in $H^1(A)$ and passing to the limits in (2.2) applied to u_n.

Proof of lemma 2.2 : "$\Leftarrow$" Fix $a_j \in \omega_j$, $j = 0, \dots, k$, and set

$$u(z) = \prod_{j=0}^{k} \left(\frac{z - a_j}{|z - a_j|} \right)^{-d_j}. \tag{A.1}$$

Then, clearly, $\deg(u, \partial\omega_j) = d_j$, $j = 0, \dots, k$, and $\deg(u, \partial\Omega) = -\sum_{j=0}^{k} d_j = D$, by (2.3), so that $u \in K$.

"$\Rightarrow$" Assume $K \neq \emptyset$ and let $u \in K$. By Lemma 2.1 and the degree formula (1.5), we have

$$\int_A \text{ Jac } u = \pi\Big(D + \sum_{j=0}^{k} d_j \Big). \tag{A.2}$$

Since $|u|^2 = 1$ a.e., we find

$$u \cdot u_x = 0 \text{ and } u \cdot u_y = 0 \text{ a.e.} \tag{A.3}$$

Hence, a.e., the vectors u_x and u_y are both orthogonal to the non-zero vector u. Therefore, $u_x \parallel u_y$ a.e., so that Jac $u = 0$ a.e.. Thus

$$\int_A \text{ Jac } u = 0 \tag{A.4}$$

and the lemma follows by combining (A.2) and (A.4).

Proof of Lemma 2.3 : a) "$\supset$" Recall that, for $s > 0$, $H^s \cap L^\infty$ is an algebra, i.e. :

(i) if $u, v \in H^s \cap L^\infty$, then $uv \in H^s \cap L^\infty$;

(ii) if $u_n, u, v_n, v \in H^s \cap L^\infty$, $u_n \to u$, $v_n \to v$ in H^s, $\|u_n\|_{L^\infty} \le C$, $\|v_n\|_{L^\infty} \le C$, then $u_n v_n \to uv$ in H^s

(see, e.g., [2]). We will also use the following well-known fact : if f is a C^1 map such that f' is bounded and if U is smooth and bounded, then, for $0 < s \le 1$, the map

$$H^s(U) \ni u \mapsto f(u) \in H^s(U)$$

is well-defined and continuous (see, e.g., [36]). It follows from the above results that the map

$$[0, 1] \ni t \mapsto v e^{\imath t \varphi} \in H^1(A; S^1) \tag{A.5}$$

is well-defined and continuous. By taking traces we find, with Γ any connected component of ∂A, that the map

$$[0, 1] \ni t \mapsto \text{ tr}_\Gamma(v e^{\imath t\varphi}) \in H^{1/2}(\Gamma; S^1) \tag{A.6}$$

is well-defined and continuous. Since the degree is continuous with respect to $H^{1/2}$ convergence, we find that deg $(\text{tr}_\Gamma(ve^{\imath\varphi}))$= deg $(\text{tr}_\Gamma(v))$. Thus $ve^{\imath\varphi} \in K$, since $v \in K$.

a) "$\subset$" Let $u \in K$ and set $w = u/v = u\overline{v}$. Since $H^1 \cap L^\infty$ is an algebra, we have $w \in H^1(A; S^1)$. We claim that deg $(w, \partial\omega_j) = 0$, $j = 0, \dots, k$. The claim is a straightforward consequence of the following

Lemma A.1. ([15]) *Let Γ be a smooth simple closed planar curve and $u, v \in H^{1/2}(\Gamma; S^1)$. Then*

$$\text{deg } (\overline{u}, \Gamma) = -\text{deg } (u, \Gamma); \tag{A.7}$$

$$\text{deg } (uv, \Gamma) = \text{deg } (u, \Gamma) + \textit{deg } (v, \Gamma); \tag{A.8}$$

$$\text{deg } (u, \Gamma) = 0 \Longleftrightarrow u = e^{\imath\varphi} \text{ for some } \varphi \in H^{1/2}(\Gamma; I\!R). \tag{A.9}$$

Proof of Lemma A.1 : All the properties are clear when u and v are smooth. In full generality, (A.7) follows from the smooth case and the continuity of the degree in $H^{1/2}$, using the fact that $C^\infty(\Gamma; S^1)$ is dense in $H^{1/2}(\Gamma; S^1)$ (see [20]). As for (A.8), it also follows by density, using in addition the fact that $H^{1/2} \cap L^\infty$ is an algebra. Implication "$\Leftarrow$" in (A.9) follows from the fact that the map

$$[0, 1] \ni t \mapsto F(t) = e^{\imath t\varphi} \in H^{1/2}(\Gamma; S^1)$$

is well-defined and continuous. Thus the degree of $F(t)$ with respect to Γ is constant, and this constant has to be 0, since $F(0)$ is a constant.
In order to prove "$\Rightarrow$" in (A.9), let U be the interior of Γ. Take a sequence $(u_n) \subset C^\infty(\Gamma; S^1)$ such that $u_n \to u$ in $H^{1/2}$ (here, we use again the density of $C^\infty(\Gamma; S^1)$ into $H^{1/2}(\Gamma; S^1)$). Since $H^{1/2} \cap L^\infty$ is an algebra, we have $v_n = u\overline{u_n} \to 1$ in $H^{1/2}$. We will make use of the following fact : there is an $\varepsilon > 0$ such that, if $v \in H^{1/2}(\Gamma; S^1)$ is such that $\|v - 1\|_{H^{1/2}} < \varepsilon$ and $\tilde{v}$ is the harmonic extension of v to U, then $1/2 \le |\tilde{v}| \le 1$ in U (see [20]). Using (A.7), (A.8), the continuity of the degree for $H^{1/2}$ convergence and the above mentioned result, we find that, for large n, we have

$$\text{deg } (u_n, \Gamma) = 0, \text{ deg } (v_n, \Gamma) = 0, \ 1/2 \le |\widetilde{v_n}| \le 1 \text{ in } U. \tag{A.10}$$

For any such n, set $w = \widetilde{v_n}/|\widetilde{v_n}|$, so that $\text{tr}_\Gamma w = \widetilde{v_n}$ and $w \in H^1(U; S^1)$. Since Γ is simple, U is simply connected. We now invoke the fact that S^1-valued H^1 maps in a simply connected domain U "lift" in H^1, i.e., we may write $w = e^{\imath\psi}$ for some $\psi \in H^1(U; I\!R)$ (see [11]). On the other hand, since deg $(u_n, \Gamma) = 0$, we may write $u_n = e^{\imath\eta}$ for some $\eta \in C^\infty(\Gamma; I\!R)$. Finally, let $\varphi = \eta +$ $\text{tr}_\Gamma\psi$. Then $\varphi \in H^{1/2}(\Gamma; I\!R)$ and clearly $u = e^{\imath\varphi}$.

Proof of Lemma 2.3 completed : Recall that deg $(w, \partial\omega_j) = 0$, $j = 0, \dots, k$. By Lemma A.1, we may thus write $w = e^{\imath\varphi_j}$ on each $\partial\omega_j$, for some $\varphi_j \in H^{1/2}(\partial\omega_j; I\!R)$. Let $\widetilde{\varphi_j}$ be the harmonic extension of φ_j to ω_j ; this $\widetilde{\varphi_j}$ belongs to $H^1(\omega_j)$. Set

$$\tilde{w} = \begin{cases} w, & \text{in A} \\ e^{\imath\widetilde{\varphi_j}}, & \text{in } \omega_j, \ j = 0, \dots, k \end{cases}.$$

Then clearly $\tilde{w} \in H^1(\Omega; S^1)$. Since Ω is simply connected, we may thus lift $\tilde{w}$ in H^1, i.e., we may write $\tilde{w} = e^{\imath\tilde{\varphi}}$ for some $\tilde{\varphi} \in H^1(\Omega; I\!R)$. If φ is the restriction of $\tilde{\varphi}$ to A, then $u = ve^{\imath\varphi}$ and $\varphi \in H^1(A; I\!R)$. This completes the proof of a).

Proof of b) : let $(u_n) \subset K$ be a bounded sequence. Write $u_n = ve^{\imath\varphi_n}$, with $\varphi_n \in H^1(A; I\!R)$. Since

$$|\nabla\varphi_n| = |\nabla(e^{\imath\varphi_n})| = |\nabla(u_n\overline{v})| = |u_n\nabla\overline{v} + \overline{v}\nabla u_n| \leq |\nabla\overline{v}| + |\nabla u_n|,$$

it follows that (φ_n) is bounded in H^1, so that, up to a subsequence, $\varphi_n \rightharpoonup \varphi$ weakly in H^1 and a.e. for some $\varphi \in H^1(A; I\!R)$. Then clearly $u_n \to u = ve^{\imath\varphi}$ a.e., so that $u_n \rightharpoonup u$ weakly in H^1. By a), this u belongs to K.

Proof of lemma 2.5 : Clearly, η is not constant. Let x_0 be a minimum point of η, which has to belong to ∂A. Let Γ be the connected component of ∂A containing x_0. Since η is constant on Γ, we must have $\dfrac{\partial\eta}{\partial\nu} < 0$ on Γ, by the Hopf boundary lemma. Therefore, $\displaystyle\int_\Gamma \frac{\partial\eta}{\partial\nu} < 0$, and thus $\Gamma = \partial\omega_0$. In conclusion, any minimum point of η is on $\partial\omega_0$. Similarly, any maximum point of η is on $\partial\Omega$, so that a) follows. For b), let $t \in (C_0, 0)$ be a regular value of η and let Γ be a connected component of the level set $\{\eta = t\}$. Note that $t \neq C_j$, $j = 1, \ldots, k$. Indeed, $\dfrac{\partial\eta}{\partial\tau} = 0$ on $\partial\omega_j$ and, if $j \geq 1$, $\dfrac{\partial\eta}{\partial\nu}$ has to vanish somewhere on $\partial\omega_j$. Thus $\nabla\eta$ has to vanish somewhere on $\partial\omega_j$, so that C_j is a critical value for $j \geq 1$. Consider the smooth domain U enclosed by Γ and let $V = U \cap A$. Then there is a family $\mathcal{F} \subset \{0, \ldots, k\}$ such that

$$\partial V = \Gamma \cup \bigcup_{j\in\mathcal{F}} \partial\omega_j.$$

Since

$$\int_{\partial V} \frac{\partial\eta}{\partial\nu} = 0 \tag{A.11}$$

and $\dfrac{\partial\eta}{\partial\nu}$ has constant sign on Γ, we see that $0 \in \mathcal{F}$. (In other words, Γ encloses $\partial\omega_0$.) Thus

$$\int_\Gamma \frac{\partial\eta}{\partial\nu} = -\int_{\partial\omega_0} \frac{\partial\eta}{\partial\nu} = 2\pi. \tag{A.12}$$

Moreover, since $\dfrac{\partial\eta}{\partial\nu}$ has constant sign on Γ, this implies that

$$\int_\Gamma |\nabla\eta| = \int_\Gamma \left|\frac{\partial\eta}{\partial\nu}\right| = 2\pi. \tag{A.13}$$

We complete the proof of b) by establishing that $\{\eta = t\} = \Gamma$. Argue by contradiction and assume that $\{\eta = t\}$ has at least two components, say Γ_1 and Γ_2. Then one of these curves must enclose the other one (since they both enclose $\partial\omega_0$). Consider the domain W contained between the two curves and let $Y = W \cap A$. Since η is constant on each component of ∂Y, it follows as above, from the Hopf boundary lemma, that η attains its maximum or minimum on Y only on $\Gamma_1 \cup \Gamma_2$. Thus η is constant in Y, which contradicts the fact that t is a regular value of η.
Finally, c) follows from Lemma 2.4 c) and the fact that

$$\int_A |\nabla\eta|^2 = \int_{\partial\omega_0} \eta\frac{\partial\eta}{\partial\nu} = -2\pi C_0. \tag{A.14}$$

Proof of Lemma 2.6 : The function η defined in b) clearly satisfies (2.13) with $C_0 = \ln\rho - \ln R$. By the uniqueness of η, this proves b) and c), while d) follows from c) and Lemma 2.5. For a), we rely on Lemma 2.4 d). We have, locally in A, $u = e^{\imath\varphi}$, where φ satisfies

$$\varphi_x = -y/|z|^2, \quad \varphi_y = x/|z|^2. \tag{A.15}$$

In the simply connected domain $U = A \setminus I\!R^-$, the solutions of (A.15) are $\theta + C$, $C \in I\!R$; here, $\theta = \theta(z) \in (-\pi, \pi)$ is the principal argumenet of z. Thus $u(z) = \alpha\dfrac{z}{|z|}$ in U, with $\alpha = e^{\imath C} \in S^1$. By continuity, we find that $u(z) = \alpha\dfrac{z}{|z|}$ in A.

Proof of Lemma 2.7 : Let v attain cap(A). Then v solves

$$\begin{cases} \Delta v = 0 & \text{in } A \\ v = 0 & \text{on } \partial\Omega \\ v = 1 & \text{on } \partial\omega_0 \end{cases} . \tag{A.16}$$

By the Hopf boundary lemma, we have $\dfrac{\partial v}{\partial\nu} > 0$ on $\partial\omega_0$. Set

$$a = \int_{\partial\omega_0} \frac{\partial v}{\partial\nu} > 0. \tag{A.17}$$

Then, clearly, the map η in Lemma 2.4 coincides with $-\dfrac{2\pi}{a}v$. Hence

$$I_0 = \frac{1}{2}\int_A |\nabla\eta|^2 = \frac{2\pi^2}{a^2}\int_A |\nabla v|^2 = \frac{2\pi^2}{a^2}\,\text{cap}(A). \tag{A.18}$$

On the other hand,

$$\operatorname{cap}(A) = \int\limits_A |\nabla v|^2 = \int\limits_{\partial\omega_0} v\frac{\partial v}{\partial \nu} = a, \tag{A.19}$$

so that the lemma follows by combining (A.18) and (A.19).

Proof of Lemma 2.8 : We start with the case of a symmetry. Without loss of generality, we may assume that $\mathcal{O}(z) = \overline{z}$. Let η be as in Lemma 2.4 and set $\tilde{\eta} = \eta \circ \mathcal{O}$. Clearly, since η satisfies (2.13), so does $\tilde{\eta}$, so that $\tilde{\eta} = \eta$, by uniqueness. Thus

$$\frac{\partial \eta}{\partial x}(\overline{z}) = \frac{\partial \eta}{\partial x}(z), \frac{\partial \eta}{\partial y}(\overline{z}) = -\frac{\partial \eta}{\partial y}(z), \quad \forall\ z \in A. \tag{A.20}$$

By Lemma 2.4 d), e), it follows that, if u is a minimizer of (1.14)-(1.15) and we write locally $u = e^{\imath\varphi}$, then

$$\frac{\partial \varphi}{\partial x}(\overline{z}) = -\frac{\partial \varphi}{\partial x}(z), \frac{\partial \varphi}{\partial y}(\overline{z}) = \frac{\partial \varphi}{\partial y}(z), \quad \forall\ z \in A. \tag{A.21}$$

Fix now a point $z_0 \in A \cap I\!\!R$ and let u be a minimizer of (1.14)-(1.15). Set $u_0 = \overline{u(z_0)}u$, so that u_0 is still a minimizer of (1.14)-(1.15) and $u_0(z_0) = 1$. Consider a ball $B \subset A$ centered at z_0 and let $J = B \cap I\!\!R$. We may write, globally in B, $u_0 = e^{\imath\varphi}$, with $\varphi(z_0) = 0$. By (A.21), we have $\dfrac{\partial \varphi}{\partial x} = 0$ on J, so that $\varphi = 0$ on J. Using again (A.21), we find that $\varphi(\overline{z}) = -\varphi(z)$ in B. In other words, the maps u_0 and $z \mapsto \overline{u_0}(\overline{z})$ coincide in B. It turns out that u_0 is analytic. Indeed, $u_0 = f/|f|$, where $f = e^{\eta}u_0$ is a holomorphic map, by (2.10). Thus, the analytic maps u_0 and $z \mapsto \overline{u_0}(\overline{z})$, that coincide in B, must coincide in A. In other words, $u_0(\overline{z}) = \overline{u_0}(z)$ in A. This is the desired symmetry property of u_0.

We now turn to the case where $\mathcal{O}$ is a rotation of angle $\theta = 2\pi/n$. Without loss of generality, we may assume the rotation centered at the origin. As above, we have, with $\tilde{\eta} = \eta \circ \mathcal{O}$, that $\tilde{\eta} = \eta$. It follows that

$$\nabla\eta(\mathcal{O}(z)) = \mathcal{O}(\nabla\eta(z)), \quad \forall\ z \in A, \tag{A.22}$$

and consequently, if we write locally $u = e^{\imath\varphi}$, then

$$\nabla\varphi(\mathcal{O}(z)) = \mathcal{O}(\nabla\varphi(z)), \quad \forall\ z \in A. \tag{A.23}$$

Fix now a smooth simple curve $\Gamma \subset A$ which encloses 0 and is symmetric with respect to $\mathcal{O}$. Such curves exist : for example, if $\delta > 0$ is sufficiently small, we may take

$$\Gamma = \{z \in A\,;\ \text{dist}\ (z, \partial\Omega) = \delta\}.$$

Note that, if we orient Γ with the natural orientation, then deg $(u, \Gamma) = 1$.

Set $v = u\dfrac{|z|}{z}$. Since deg $(v, \Gamma) = 0$, we may write $v = e^{\imath\psi}$ on Γ, for some smooth globally defined ψ. By (A.23), with $\tau(z)$ the direct tangent vector to Γ at z, we have

$$\tau(\mathcal{O}(z)) \cdot \nabla\psi(\mathcal{O}(z)) = \tau(z) \cdot \nabla\psi(z), \quad \forall\ z \in \Gamma. \tag{A.24}$$

It follows that, for some $C \in I\!R$, we have

$$\psi(\mathcal{O}(z)) = \psi(z) + C, \quad \forall\ z \in \Gamma. \tag{A.25}$$

Therefore,

$$\psi(z) = \psi(\mathcal{O}^n(z)) = \psi(z) + nC, \quad \forall\ z \in \Gamma, \tag{A.26}$$

so that $C = 0$. By (A.23), we then have $v(\mathcal{O}(z)) = (v(z))$ on Γ. Going back to u, we find that

$$u(\mathcal{O}(z)) = \mathcal{O}(u(z)), \quad \forall\ z \in \Gamma. \tag{A.27}$$

Finally, set $f = e^{\eta}u$, which is holomorphic in A. Using (A.27) and the fact that $\tilde{\eta} = \eta$, we find that the holomorphic maps f and $\mathcal{O}^{-1} \circ f \circ \mathcal{O}$ coincide on Γ. Thus they must coincide in A. This proves that $u \circ \mathcal{O} = \mathcal{O} \circ u$ for every minimizer of (1.14)-(1.15).

The last possible case is $k = 0$ and $\mathcal{O}$ rotation of angle θ, with $\theta \notin \pi\mathbb{Q}$. But then A has to be a circular annulus, and in this case the conclusion follows from Lemma 2.6.

Appendix B. Zeroes of complex valued maps

We present below an analogue of the property f) of the degree mentioned in the Introduction, in case where we consider maps which are not smooth up to the boundary. For a more refined statement, see [20].

Lemma B.1. *Let $u \in H^1(A; C)$ be such that $\Delta u \in L^2$. Assume that $1/2 \le |u| \le 1$ on ∂A and that*

$$\deg(u/|u|, \partial\Omega) + \sum_{j=0}^{k} \deg(u/|u|, \partial\omega_j) \neq 0. \tag{B.1}$$

Then u vanishes at least once in A.

Proof : Argue by contradiction. We first claim that there is some $a > 0$ such that $1/a \le |u| \le a$ in A. Indeed, write $u = v + w$, where v is the hamonic extension of tr u. Since $\Delta w \in L^2$, we find that w is continuous in $\overline{A}$ (and vanishes on ∂A). On the other hand, tr v takes its values into the closed set $F = \{z \in \mathbb{C}\ ;\ 1/2 \le |z| \le 1\}$. Therefore, when z is sufficiently close to ∂A, $v(z)$ is close to F :

$$\exists\ \delta > 0 \text{ such that } \mathrm{dist}(z, \partial A) < \delta \Longrightarrow 1/4 \le |v(z)| \le 2\ ; \tag{B.2}$$

this property was established in [20] for harmonic extensions of VMO maps. In our case, tr $v \in H^{1/2}(\partial A)$, and $H^{1/2} \hookrightarrow$VMO in 1D ; therefore, (B.2) applies to our case. We find that $1/8 \le |u(z)| \le 4$ if z is sufficiently close to ∂A. Finally, the existence of a is a consequence of the continuity of u in A.
Consider the map $U = u/|u|$; this map is well-defined, by the preceding discussion. On the other hand, since U is S^1-valued, Lemma 2.2 implies that

$$\deg(u/|u|, \partial\Omega) + \sum_{j=0}^{k} \deg(u/|u|, \partial\omega_j) = 0. \tag{B.3}$$

This contradicts assumption (B.1) and completes the proof of Lemma B.1.

Appendix C. Smoothness of critical points

This appendix is devoted to the

Proof of Lemma 4.4 : For the convenience of the reader, we divide the proof, which is rather long and technical, into several steps.
Step 1. We have $|v_\kappa(z)| \to 1$ uniformly as $\mathrm{dist}(z, \partial A) \to 0$
Let $g = \mathrm{tr}_{\partial A} v_\kappa \in H^{1/2}(\partial A; S^1)$. Since in particular v_κ is a critical point of the Ginzburg-Landau energy in the class

$$\mathcal{L} = \{v \in H^1(A; \mathbb{C});\ \mathrm{tr}_{\partial A} v = g\}, \tag{C.1}$$

v_κ satisfies the first equation in (4.15). Property c) follows from the first equation in (4.15), the maximum principle and the fact that $|g| = 1$ (see [9]). By a standard bootstrap argument, the first equation in (4.15) also implies that $v_\kappa \in C^\infty(A)$. The non trivial assertion of the Lemma is smoothness up to the boundary, and the remaining part of the proof is devoted to establishing this assertion.
We split $v_\kappa = w + \tilde{g}$, where w and $\tilde{g}$ solve respectively

$$\begin{cases} -\Delta w &= \kappa^2 v_\kappa (1 - |v_\kappa|^2) \quad \text{in } A \\ \quad w &= 0 \quad \text{on } \partial A \end{cases} \tag{C.2}$$

and

$$\begin{cases} -\Delta \tilde{g} &= 0 \quad \text{in } A \\ \quad \tilde{g} &= g \quad \text{on } \partial A \end{cases} . \tag{C.3}$$

Since $H^1(A) \hookrightarrow L^q(A)$, $1 \le q < \infty$, we find that $\Delta w \in L^p(A)$, for $1 < p < \infty$. Thus $w \in W^{2,p}(A)$, $1 < p < \infty$, by standard elliptic estimates (see, e.g., [26]). In particular, by the Sobolev embeddings we have $w \in C^{1,\alpha}(\overline{A})$ for $0 < \alpha < 1$. Since $w = 0$ on ∂A, this implies that

$$|w(z)| \le C\ \mathrm{dist}(z, \partial A), \quad \forall\ z \in A. \tag{C.4}$$

On the other hand, given a map $g \in \mathrm{VMO}(\partial A\,; S^1)$, its harmonic extensions to A, $\tilde{g}$, has modulus "almost" 1 near ∂A, that is

$$g \in \mathrm{VMO}(\partial A\,; S^1), 0 < \varepsilon < 1 \Longrightarrow \exists\, \delta > 0 \text{ such that } 1-\varepsilon \leq |\tilde{g}(z)| \leq 1 \text{ if } \operatorname{dist}(z, \partial A) < \delta \quad \text{(C.5)}$$

(for the definition of VMO and the proof of (C.5), see [20]). Since $H^{1/2} \hookrightarrow$VMO in 1D, the conclusion of (C.5) holds for our g. By combining (C.4) and (C.5), we find that

$$|v_\kappa(z)| \to 1 \text{ uniformly as } \operatorname{dist}(z, \partial A) \to 0. \quad \text{(C.6)}$$

Step 2. We rewrite the Ginzburg-Landau equation near ∂A

We fix a neighborhood U of ∂A such that $|v_\kappa| \geq 1/2$ in $U \cap A$. From now on, we work only in $V = U \cap A$. In V we may write $v_\kappa = \rho\eta$, where $\rho = |v_\kappa|$ and $\eta = \dfrac{v_\kappa}{|v_\kappa|}$. Since $\eta \in C^\infty(V; S^1)$ we may write, in simply connected subdomains of V, $\eta = e^{\imath\psi}$. The vector field $\nabla\psi = \eta \wedge \nabla\eta$ is globally defined in V, and clearly $\nabla\psi \in L^2(V)$, since $\eta \in H^1(V)$. It is easy to see that ρ and ψ are weak solutions of

$$\begin{cases} -\Delta\rho &= \kappa^2\rho(1-\rho^2) - \rho|\nabla\psi|^2 \quad \text{in } V \\ \quad\ \rho &= 1 \quad \text{on } \partial A \end{cases} \quad \text{(C.7)}$$

and respectively

$$\begin{cases} -\operatorname{div}(\rho^2\nabla\psi) &= 0 \quad \text{in } V \\ \quad \nu\cdot\nabla\psi &= 0 \quad \text{on } \partial A \end{cases}. \quad \text{(C.8)}$$

The last condition in (C.8) is obtained via the fact that

$$\left[\frac{d}{dt}E_\kappa(v_\kappa e^{\imath t\zeta})\right]_{t=0} = 0, \quad \forall\, \zeta \in C^\infty(\overline{A}) \text{ such that } \operatorname{supp}\zeta \subset \overline{V}. \quad \text{(C.9)}$$

We are eventually going to prove that ρ, ψ are smooth in the neighborhood of each point $z_0 \in \partial A$, which will complete the proof of the Lemma.

Step 3. We have $\nabla\psi \in L^p(V)$, $1 \leq p < \infty$

Fix some $z_0 \in \partial A$. In order to simplify the proof, we make the following assumptions : we suppose that $z_0 = 0$, that $A \subset \{z\,; \mathrm{Im}(z) > 0\}$, and that $\partial A \subset I\!R$ in the neighborhood of z_0. (However, these assumptions are not essential for carrying out the arguments below.) Let $R > 0$ to be specified later such that the disc D_R of radius R is contained in U and the upper half disc $D_R^+ = D_R \cap \{z\,; \mathrm{Im}(z) > 0\}$ is contained in V. We extend ψ, ρ and $F = (1-\rho^2)\nabla\psi = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ from the upper half disc D_R^+ to D_R. These extensions will be denoted by $\tilde{\psi}$, $\tilde{\rho}$ and $\tilde{F}$ and are given, in the lower half disc $D_R \setminus \overline{A}$, by

$$\tilde{\psi}(z) = \psi(\overline{z}),\ \tilde{\rho}(z) = \rho(\overline{z}),\ \tilde{F}(z) = \begin{pmatrix} F_1(\overline{z}) \\ -F_2(\overline{z}) \end{pmatrix}. \quad \text{(C.10)}$$

Then, clearly, $\tilde{\psi}$ is a weak solution of

$$\Delta\tilde{\psi} = \operatorname{div}\tilde{F}(z) \quad \text{ in } D_R. \tag{C.11}$$

By standard elliptic estimates ([26]), we have

$$\|\nabla\tilde{\psi}\|_{L^p(D_R)} \leq C_p(\|\mathrm{tr}_{\partial D_R}\tilde{\psi}\|_{W^{1-1/p,p}(\partial D_R)} + \|\tilde{F}\|_{L^p(D_R)}),\ \forall\ 1 < p < \infty. \tag{C.12}$$

By scaling, the constant C_p depends on p, but not on R. Next we note that

$$\|\tilde{F}\|_{L^p(D_R)} \leq \|1-\tilde{\rho}^2\|_{L^\infty(D_R)}\|\nabla\tilde{\psi}\|_{L^p(D_R)} = \|1-\rho^2\|_{L^\infty(D_R)}\|\nabla\tilde{\psi}\|_{L^p(D_R)} \leq \frac{1}{2C_p}\|\nabla\tilde{\psi}\|_{L^p(D_R)}, \tag{C.13}$$

provided R is sufficiently small, by (C.6) and the definition of $\tilde{\rho}$. We next note that, for a.e. $R > 0$ such that $D_R^+ \subset V$, we have $\mathrm{tr}_{\partial D_R \cap A}\psi \in H^1(\partial D_R \cap A)$, so that for any such R we have

$$\mathrm{tr}_{\partial D_R}\tilde{\psi} \in H^1(\partial D_R) \hookrightarrow W^{1-1/4,4}(\partial D_R), \tag{C.14}$$

by the Sobolev embeddings. Taking any R such that both (C.13) with $p = 4$ and (C.14) hold, we find from (C.12) that $\nabla\psi \in L^4$ near z_0. Using the fact that $v_\kappa \in C^\infty(A)$, we obtain that $\nabla\psi \in L^4(V)$. We use this argument to bootstrap : as above, for a.e. $R > 0$ such that $D_R^+ \subset V$ we have

$$\mathrm{tr}_{\partial D_R}\psi \in W^{1,4}(\partial D_R) \hookrightarrow W^{1-1/8,8}(\partial D_R). \tag{C.15}$$

We find similarly that $\nabla\psi \in L^8(V)$, and by induction that $\nabla\psi \in L^{2^n}(V)$ for $n \geq 1$. Therefore,

$$\nabla\psi \in L^p(V) \text{ for } 1 \leq p < \infty. \tag{C.16}$$

Step 4. The bootstrap argument
Going back to the equation (C.7), we find from (4.31) that $\Delta\rho \in L^p(V)$, $1 < p < \infty$. If W is a neighborhood of ∂A such that $\overline{W} \subset U$, this implies, by standard elliptic estimates ([26]), that $\rho \in W^{2,p}(W \cap A)$, $1 < p < \infty$, and therefore,

$$\rho \in W^{2,p}(V),\ 1 < p < \infty, \tag{C.17}$$

since $v_\kappa \in C^\infty(A)$.
Finally, we bootstrap the equation (C.7) and the equation (C.8) of ψ. For this purpose, we pick, for each $z_0 \in \partial A$, a disc D centered at z_0 such that $D \cap V$ is simply connected. We may thus choose, in $D \cap V$, a single-valued phase ψ of v_κ. Then, in $D \cap V$, we may rewrite (C.8) as

$$\begin{cases} \Delta\psi & = -\dfrac{2}{\rho}\nabla\rho\cdot\nabla\psi \quad \text{in } D \cap V \\ \dfrac{\partial\psi}{\partial\nu} & = 0 \qquad\qquad\qquad \text{on } \partial A \cap D \end{cases}. \tag{C.18}$$

From (C.16)-(C.18), we find that $\Delta\psi \in L^p(D\cap V)$, $1<p<\infty$. This implies that $\psi \in W^{2,p}(V)$, $1<p<\infty$. We next obtain by a straightforward induction that $\rho \in W^{n,p}(V)$, $\psi \in W^{n,p}(V)$, $1<p<\infty$, $n\geq 2$. Since $v_\kappa = \rho e^{\imath\psi}$ in $D\cap V$, we find that $v_\kappa \in C^\infty(\overline{D\cap V})$. Thus $v_\kappa \in C^\infty(\overline{A})$ and the proof of Lemma 4.4 is complete.

Appendix D. On the harmonic extension in a circular annulus

Here, we make explicit some straightforward and well known computations we needed in Section 5. Throughout this Appendix, we asume that A is a circular annulus, $A=\{z\,;\,\rho<|z|<R\}$. Let $g\in H^{1/2}(A)$. We may thus write, on $\partial\omega_0=\{z\,;\,|z|=\rho\}$, $g=\sum\limits_{n\in\mathbb{Z}} a_n e^{\imath n\theta}$; since $g\in H^{1/2}$, we have $\sum\limits_{n\in\mathbb{Z}}|n||a_n|^2<\infty$. Similarly, on $\partial\Omega=\{z\,;\,|z|=R\}$, we may write $g=\sum\limits_{n\in\mathbb{Z}} b_n e^{\imath n\theta}$, with $\sum\limits_{n\in\mathbb{Z}}|n||b_n|^2<\infty$.

Lemma D.1. *Let u be the harmonic extension of g to A. Then*

$$\int_A |\nabla u|^2 = 2\pi\Big\{\frac{|b_0-a_0|^2}{\ln R-\ln\rho}+\sum_{n\neq 0}\frac{|n|}{R^{2|n|}-\rho^{2|n|}}[(|a_n|^2+|b_n|^2)(R^{2|n|}+\rho^{2|n|})-2(\overline{a_n}b_n+a_n\overline{b_n})R^{|n|}\rho^{|n|}]\Big\}. \tag{D.1}$$

Proof : Since u is harmonic in A, we may write in polar coordinates

$$u=A_0+B_0\ln r+\sum_{n\neq 0}(A_n r^{|n|}+B_n r^{-|n|})e^{\imath n\theta}\equiv\sum\alpha_n e^{\imath n\theta}. \tag{D.2}$$

Identification of the coefficients on ∂A yields

$$A_0=\frac{a_0\ln R-b_0\ln\rho}{\ln R-\ln\rho},\quad B_0=\frac{b_0-a_0}{\ln R-\ln\rho} \tag{D.3}$$

and

$$A_n=\frac{b_nR^{|n|}-a_n\rho^{|n|}}{(R^{|n|}-\rho^{|n|})(R^{|n|}+\rho^{|n|})},\quad B_n=\frac{R^{|n|}\rho^{|n|}(a_nR^{|n|}-b_n\rho^{|n|})}{(R^{|n|}-\rho^{|n|})(R^{|n|}+\rho^{|n|})},\quad\forall\ n\neq 0. \tag{D.4}$$

Since clearly

$$\int_A|\nabla u|^2=2\pi\int_\rho^R(r\sum|\alpha_n{}'|^2+\frac{1}{r}|\alpha_n|^2), \tag{D.5}$$

inserting (D.3) and (D.4) into (D.5) yields (D.1).

Remark D.1. It follows from Lemma D.1 that the right-hand side of (D.1) yields a "standard" semi-norm on $H^{1/2}(\partial A)$, in the following sense : this quantity is a norm, equivalent to the usual ones, on codimension one subspaces of $H^{1/2}(\partial A)$ that do not contain non zero constants. For example, a possible choice would be $\{g \in H^{1/2}(\partial A)\,;\ \int\limits_{\partial\Omega} g = 0\}$. Moreover, for any standard norm on $H^{1/2}(\partial A)$, we have

$$C_1 \Big\| g - \frac{1}{2\pi R} \int\limits_{\partial\Omega} g \Big\|^2_{H^{1/2}(\partial A)} \leq \text{r.h.s. of (D.1)} \leq C_2 \Big\| g - \frac{1}{2\pi R} \int\limits_{\partial\Omega} g \Big\|^2_{H^{1/2}(\partial A)} \tag{D.6}$$

for some $C_1 > 0$, $C_2 > 0$ independent of g.

Lemma D.2. *There are constants $C_1 > 0$, $C_2 > 0$ such that*

$$C_1\Big(|b_0 - a_0|^2 + \sum_{n\neq 0} |n|(|a_n|^2+|b_n|^2)\Big) \leq \text{r.h.s. of (D.1)} \leq C_1\Big(|b_0 - a_0|^2 + \sum_{n\neq 0} |n|(|a_n|^2+|b_n|^2)\Big). \tag{D.7}$$

Proof : In view of (D.1), it suffices to establish, for $n \neq 0$, with some constants C_1, C_2 independent of n, a_n, b_n, the inequality

$$C_1(|a_n|^2+|b_n|^2) \leq \frac{1}{R^{2|n|} - \rho^{2|n|}}[(|a_n|^2+|b_n|^2)(R^{2|n|}+\rho^{2|n|}) - 2(\overline{a_n}b_n + a_n\overline{b_n})R^{|n|}\rho^{|n|}] \leq C_2(|a_n|^2+|b_n|^2). \tag{D.8}$$

On the one hand, we have

$$(|a_n|^2+|b_n|^2)(R^{2|n|}+\rho^{2|n|}) - 2(\overline{a_n}b_n + a_n\overline{b_n})R^{|n|}\rho^{|n|} \geq (|a_n|^2+|b_n|^2)(R^{2|n|}+\rho^{2|n|}) - 2(|a_n|^2+|b_n|^2)R^{|n|}\rho^{|n|}, \tag{D.9}$$

so that

$$(|a_n|^2+|b_n|^2)(R^{2|n|}+\rho^{2|n|}) - 2(\overline{a_n}b_n + a_n\overline{b_n})R^{|n|}\rho^{|n|} \geq (|a_n|^2+|b_n|^2)(R^{|n|}-\rho^{|n|})^2, \tag{D.10}$$

and thus

$$\frac{1}{R^{2|n|} - \rho^{2|n|}}[(|a_n|^2+|b_n|^2)(R^{2|n|}+\rho^{2|n|}) - 2(\overline{a_n}b_n + a_n\overline{b_n})R^{|n|}\rho^{|n|}] \geq \frac{R-\rho}{R+\rho}(|a_n|^2+|b_n|^2)\,; \tag{D.11}$$

here, we use the fact that $\dfrac{R^{|n|} - \rho^{|n|}}{R^{|n|} + \rho^{|n|}} \geq \dfrac{R-\rho}{R+\rho}$.

Similarly, starting from

$$(|a_n|^2+|b_n|^2)(R^{2|n|}+\rho^{2|n|}) - 2(\overline{a_n}b_n + a_n\overline{b_n})R^{|n|}\rho^{|n|} \leq (|a_n|^2+|b_n|^2)(R^{2|n|}+\rho^{2|n|}) + 2(|a_n|^2+|b_n|^2)R^{|n|}\rho^{|n|}, \tag{D.12}$$

we obtain

$$\frac{1}{R^{2|n|}-\rho^{2|n|}}[(|a_n|^2+|b_n|^2)(R^{2|n|}+\rho^{2|n|})-2(\overline{a_n}b_n+a_n\overline{b_n})R^{|n|}\rho^{|n|}]\le\frac{R+\rho}{R-\rho}(|a_n|^2+|b_n|^2), \quad \text{(D.13)}$$

since $\dfrac{R^{|n|}+\rho^{|n|}}{R^{|n|}-\rho^{|n|}}\ge\dfrac{R+\rho}{R-\rho}$.
The conclusion of Lemma D.2 follows from (D.11) and (D.13).

Corollary D.1. *The map*

$$H^{1/2}(\partial A)\ni g\mapsto|g|^2_{H^{1/2}(\partial A)}=|b_0-a_0|^2+\sum_{n\neq0}|n|(|a_n|^2+|b_n|^2) \quad \text{(D.14)}$$

is a norm equivalent with the usual ones on $H^{1/2}(\partial A)$ modulo constants.

Lemma D.3. *Let $A=\{z\,;\,\rho<|z|<R\}$. Then :*
a) $m_0=2\pi\dfrac{R-\rho}{R+\rho}$;
b) the only minimizers of (1.1)-(1.3) for $\kappa=0$ are, in polar coordinates, of the form $u=\alpha\dfrac{r^2+R\rho}{r(R+\rho)}e^{\imath\theta}$.

Proof : Clearly, the map given in b) belongs to $\mathcal{J}$ and is the harmonic extension to A of $g:\partial A\to\mathbb{C}$, $g(z)=\alpha\dfrac{z}{|z|}$. Moreover, by (D.1), the Dirichlet integral of this u is given by $\dfrac{1}{2}\int\limits_A|\nabla u|^2=2\pi\dfrac{R-\rho}{R+\rho}$. Therefore, it suffices to prove that the only minimizers of (1.1)-(1.3) are those given in b). Let v in $\mathcal{J}$. Let $g\in H^{1/2}(\partial A)$ be the trace of v to ∂A and let u be the harmonic extension to A. The starting point is (D.10), that yields, after substitution in (D.1),

$$\int\limits_A|\nabla u|^2\ge2\pi\Big\{\frac{|b_0-a_0|^2}{\ln R-\ln\rho}+\sum_{n\neq0}\frac{|n|(R^{|n|}-\rho^{|n|})}{R^{|n|}+\rho^{|n|}}(|a_n|^2+|b_n|^2)\Big\}. \quad \text{(D.15)}$$

Equality in (D.15) requires equality in (D.10), which holds if and only if

$$a_n=b_n \quad \text{for } n\neq0. \quad \text{(D.16)}$$

It follows from (D.15) that

$$\int\limits_A|\nabla u|^2\ge2\pi\sum_{n>0}\frac{n(R-\rho)}{R+\rho}(|a_n|^2+|b_n|^2), \quad \text{(D.17)}$$

and equality in (D.17) holds if and only if

$$a_n = b_n = 0 \quad \text{for } n \neq 1. \tag{D.18}$$

Since $v \in \mathcal{J}$, we have $\sum n|a_n|^2 = \sum n|b_n|^2 = 1$, by the degree formula (1.6). Thus, we finally obtain

$$\int\limits_A |\nabla v|^2 \geq \int\limits_A |\nabla u|^2 \geq 2\pi \sum_{n>0} \frac{n(R-\rho)}{R+\rho}(|a_n|^2 + |b_n|^2) \geq 2\pi \frac{R-\rho}{R+\rho} \sum n(|a_n|^2 + |b_n|^2) = 4\pi \frac{R-\rho}{R+\rho}, \tag{D.19}$$

with equality if and only if

$$a_n = b_n = 0 \quad \text{for } n < 0 \tag{D.20}$$

and

$$v \text{ is the harmonic extension of } g. \tag{D.21}$$

Therefore, equality in (D.19) holds if and only if v is the harmonic extension of $\alpha \dfrac{z}{|z|}$, i.e., if and only if v is as in b).

Appendix E. Elementary estimates for conformal mappings

Lemma E.1. *Let a, $w \in \mathbf{D}$. Then*

$$|1 - \bar{a}w| \leq 1 - |a|^2 + |w - a|, \tag{E.1}$$

$$|1 - \bar{a}w|^2 = |w - a|^2 + (1 - |a|^2)(1 - |w|^2), \tag{E.2}$$

$$\left|\frac{w-a}{1-\bar{a}w}\right| \leq \frac{|w| + |a|}{1 + |a||w|}, \tag{E.3}$$

$$\left|\frac{w-a}{1-\bar{a}w}\right| \geq \frac{||w| - |a||}{1 - |a||w|}. \tag{E.4}$$

Proof. We have

$$\begin{aligned} |1 - \bar{a}w| &= |1 - a\bar{a} + a\bar{a} - \bar{a}w| \leq 1 - |a|^2 + |a\bar{a} - \bar{a}w| \\ &= 1 - |a|^2 + |a||w - a| \leq 1 - |a|^2 + |w - a|, \end{aligned}$$

which proves (E.1). The identity (E.2) can be easily checked.

We finally prove (E.3). The proof of (E.4) being identical to the one of (E.3). Firstly, we may assume $a \geq 0$, since $\left|\frac{w-a}{1-\bar{a}w}\right| = \left|\frac{e^{i\theta}w - e^{i\theta}a}{1-\overline{e^{i\theta}a}e^{i\theta}w}\right|$,and the conclusion depends only on $|a|$, $|w|$.

Then, squaring (E.3), we find that

$$\begin{aligned} (x3) \iff \; & +a(w + \bar{w})(1 - |w|^2)(1 - a^2) \\ & \geq (|w|^2 + a^2)(1 + a|w|)^2 - (|w| + a)^2(1 + |a|^2w^2). \end{aligned} \tag{E.5}$$

Clearly, for $|w| = \rho$ fixed, the left hand side of (E.5) is minimal when $w + \bar{w}$ is minimal, i.e. for $w = -\rho$. Thus

$$\mathrm{L}HS \quad of \quad (x5) \geq -2a|w|(1-|w|^2)(1-a^2), \tag{E.6}$$

and it is easy to see that the RHS of (E.6) coincides with the RHS of (E.5).

Lemma E.2. *Let $r \in (0,1)$ be fixed. If*

$$\left|\frac{w-a}{1-\bar{a}w}\right| \leq r, \tag{H}$$

then

$$|w - a \leq \frac{2r}{1-r}(1-|a|), \tag{E.7}$$

$$1-|w| \geq \frac{1-r}{1+r}(1-|a|). \tag{E.8}$$

Proof. We have, by (E.1) and (H)

$$r \geq \left|\frac{w-a}{1-\bar{a}w}\right| \geq \frac{|w-a|}{1-|a|^2+|w-a|}.$$

Thus

$$(1-r)|w-a| \leq r(1-|a|^2) \leq 2r(1-|a|),$$

so that (E.7) follows.

On the other hand (E.4) and (H) yield

$$r \geq \frac{|w-a|}{|1-\bar{a}w|} \geq \frac{||w|-|a||}{1-|\bar{a}||\bar{w}|} \geq \frac{|w|-|a|}{1-|a||w|},$$

so that $(1 + r|a|)|w| \leq r + |a|$, i.e.,

$$1-|w| \geq 1 - \frac{r+|a|}{1+r|a|} = \frac{(1-r)(1-|a|)}{1+r|a|} \geq \frac{1-r}{1+r}(1-|a|).$$

Lemma E.3. *Let $c > 0$, $k > 0$ and assume that*

$$|w-a| \leq k(1-|a|), \tag{$H1$}$$

$$1-|w| \geq c(1-|a|). \tag{$H2$}$$

Then

$$\left|\frac{w-a}{1-\bar{a}w}\right| \leq \frac{1}{\sqrt{1+\frac{c}{k^2}}}. \tag{E.9}$$

Proof. Using (E.2), (H1) and (H2), we find that

$$\left|\frac{w-a}{1-\bar{a}w}\right| = \frac{1}{\sqrt{1+\frac{(1-|w|^2)(1-|a|^2)}{|w-a|^2}}}$$

$$\leq \frac{1}{\sqrt{1+\frac{(1-|w|)(1-|a|)}{|w-a|^2}}} \leq \frac{1}{\sqrt{1+\frac{c(1-|a|)^2}{k^2(1-|a|)^2}}} = \frac{1}{\sqrt{1+\frac{c}{k^2}}}.$$

Lemma E.4. *Let Ω be a smooth bounded simply connected domain in $\mathbf{R}^2$ and let $\phi\colon \Omega \to \mathbf{D}$ be a conformal representation. Then, with constants G depending only on ϕ, we have*

$$C_1|\zeta_1-\zeta_2| \leq |\phi(\zeta_1)-\phi(\zeta_2)| \leq C_2|\zeta_1-\zeta_2|, \quad \forall \zeta_1,\zeta_2 \in \Omega, \tag{E.10}$$

$$C_1 \mathrm{dist}(\zeta,\partial\Omega) \leq 1-|\phi(\zeta)| \leq C_2 \mathrm{dist}(\zeta,\partial\Omega), \quad \forall \zeta \in \Omega. \tag{E.11}$$

Proof. (E.10) is a trivial consequence of the fact that ϕ extends as a diffeomorphism of $\bar{\Omega}$ into $\bar{\mathbf{D}}$. as for (E.11), on the one hand we have

$$1-|\phi(\zeta)| \leq |w-\phi(\zeta)|, \quad \forall w \in S^1.$$

Thus

$$1-|\phi(\zeta)| \leq |\phi(\phi^{-1}(w))-\phi(\zeta)| \leq C_2|\phi^{-1}(w)-\zeta|,$$

so that

$$1-|\phi(\zeta)| \leq C_2 \inf_{w\in S^1}|\phi^{-1}(w)-\zeta| = C_2 \inf_{\xi\in\partial\Omega}|\xi-\zeta| = C_2 \mathrm{dist}(\zeta,\partial\Omega).$$

On the other hand,

$$1-|\phi(\zeta)| = \mathrm{dist}(\phi(\zeta),S^1) = \inf_{w\in S^1}|\phi(\zeta)-w| = \inf_{w\in S^1}|\phi(\zeta)-\phi(\phi^{-1}(w))|$$

$$\geq C_1 \inf_{w\in S^1}|\zeta-\phi^{-1}(w)| = C_1 \mathrm{dist}(\zeta,\partial\Omega).$$

Lemma E.5. *Set, for $\xi \in \Omega$ and $\alpha \in S^1$,*

$$\phi_{\eta,\alpha}(\zeta) = \alpha\frac{\phi(\zeta)-\phi(\eta)}{1-\bar{\phi}(\eta)\phi(\zeta)}$$

(these are all the conformal mappings of Ω into $\mathbf{D}$ vanishing at η).

a) Fix some $r \in (0,1)$. Then there are constants k_1, $k_2 \geq 0$ independent of α, η such that

$$\phi_{\eta,\alpha}^{-1}(\bar{\mathbf{D}}_r) \subset \{\zeta \in \Omega; |\zeta-\eta| \leq k_1 \mathrm{dist}(\eta,\partial\Omega), \mathrm{dist}(\zeta,\partial\Omega) \geq k_2 \mathrm{dist}(\eta,\partial\Omega)\}.$$

b) Fix some k_1, $k_2 > 0$. Then there is some $r \in (0,1)$ independent of α, η such that

$$\phi_{\eta,\alpha}^{-1}(\bar{\mathbf{D}}_r) \supset \{\zeta \in \Omega; |\zeta-\eta| \leq k_1 \mathrm{dist}(\eta,\partial\Omega), \mathrm{dist}(\zeta,\partial\Omega) \geq k_2 \mathrm{dist}(\eta,\partial\Omega)\}.$$

Proof. We may clearly assume $\alpha = 1$. Set $a = \phi(\eta)$, $w = \phi(\zeta)$. we have

$$\zeta \in \phi_{\eta,1}^{-1}(\bar{\mathbf{D}}_r) \iff \phi_{\eta,1}(\zeta) = w \in \bar{\mathbf{D}}_r \iff \left|\frac{w-a}{1-\bar{a}w}\right| \le r$$

$\Longrightarrow$ by (E.7) and (E.8)

$$|w-a| \le \frac{2r}{1-r}(1-|a|) \quad \text{and} \quad 1-|w| \ge \frac{1-r}{1+r}(1-|a|). \tag{E.12}$$

Lemma E.4 combined with (E.12) yields

$$|\zeta - \eta| \le \frac{2C_2 r}{C_1(1-r)} \text{dist}(\eta, \partial\Omega) \tag{E.13}$$

and

$$\text{dist}(\zeta, \partial\Omega) \ge \frac{C_1(1-r)}{C_2(1+r)} \text{dist}(\xi, \partial\Omega), \tag{E.14}$$

which proves a).

Lemma E.4 implies that

$$M = \{\zeta \in \Omega; |\zeta - \eta| \le k_1 \text{dist}(\eta, \partial\Omega), \text{dist}(\zeta, \partial\Omega) \ge k_2 \text{dist}(\eta, \partial\Omega)\}$$

$$\subset N = \{\zeta \in \Omega; |\phi(\zeta) - \phi(\eta)| \le k_1 \frac{C_1}{C_2}(1-|\phi(\eta)|) \quad \text{and} \quad 1-|\phi(\zeta)| \ge k_2 \frac{C_1}{C_2}(1-|\phi(\eta)|)\}.$$

For $\zeta \in N$, Lemma E.3 implies that

$$|\phi_{\eta,1}(\zeta)| = \left|\frac{w-a}{1-\bar{a}w}\right| \le \frac{1}{\sqrt{1+\dfrac{k_2 C_1^3}{k_1^2 C_2^3}}} = r < 1,$$

that is

$$\phi_{\eta,1}(M) \subset \phi_{\eta,1}(N) \subset \bar{\mathbf{D}}_r.$$

13 Update

A short version of this preprint appeared as
Leonid Berlyand, Petru Mironescu, *Ginzburg-Landau minimizers with prescribed degrees. Capacity of the domain and emergence of vortices*, Journal of Functional Analysis 239 (2006), 76–99.
This paper contains essentialy the proof of Theorem 1 and a soft version of the estimates in Section 11.
The main question left open in this preprint is non existence, for large κ, of minimizers of E_κ when

A has a single hole and is supercritical. This was (positively) answered in the note
Leonid Berlyand, Dmitry Golovaty, Volodymyr Rybalko, *Nonexistence of Ginzburg-Landau minimizers with prescribed degree on the boundary of a doubly connected domain*, Comptes Rendus.Mathématique 343 (2006), 63–68.
The proof there requires new ingredients. It uses also the estimates in Section 11.1. A shortcut in obtaining these estimates can be find in
Leonid Berlyand, Petru Mironescu, *Two-parameter homogenization for a Ginzburg-Landau problem in a perforated domain* , posted at http://hal.archives-ouvertes.fr/ICJ/
Among other results, it is explained there that, for large κ and supercritical A, nonexistence of minimizers of E_κ holds also when A has more than one hole.
Almost nothing is known about existence (or rather nonexistence) when the degrees prescribed on ∂A are arbitrary. A natural method for proving nonexistence is to prove nonexistence of critical points, i. e., of solutions of (4.15). Bad news: in
Leonid Berlyand, Volodymyr Rybalko, *Solutions with Vortices of a Semi-Stiff Boundary Value Problem for the Ginzburg-Landau Equation*, posted at http://arxiv.org/abs/0712.1062v1
it is proved that, when A has a single hole, the prescribed degrees are arbitrary and κ is large, (4.15) has always solutions.
Mickaël Dos Santos announced the same result for an arbitrary number of holes. Result to be posted at http://hal.archives-ouvertes.fr/ICJ/

References

[1] L. Ahlfors, *Complex Analysis*, McGraw-Hill, 1966.

[2] S. Alinhac and P. Gérard, *Opérateurs pseudo-différentiels et théorème de Nash-Moser*, Interéditions, 1991.

[3] Th. Aubin, *Equations différentieles nonlinéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. , 55(1976), 269-293.

[4] L. Berlyand, *Effective properties of superconducting and superfluid composites*, Int. J. Modern Physics b, 12(1998), 3063-3073.

[5] L. Berlyand and E. Khruslov, *Homogenization of harmonic maps and superconducting composites*, SIAM J. Appl. Math., 59 (1999), 1892-1916.

[6] L. Berlyand and P. Mironescu, *Ginzburg-Landau minimizers with prescribed degrees : dependence on domain*, C. Rendus Acad. Sci. Paris337 (2003), 375-380.

[7] L. Berlyand and K. Voss, *Symmetry breaking in annular domains for a Ginzburg-Landau superconductivity model*, Proceedings of IUTAM 99/4 Symposium (Sydney, Australia), Kluwer Academic Publishers, 1999.

[8] F. Bethuel, H. Brezis, B. D. Coleman and F. Hélein, *Bifurcation analysis of minimizing harmonic maps describing the equilibrium of nematic phases between cylinders*, Arch. Rational Mech. Anal., 118 (1992), 149-168.

[9] F. Bethuel, H. Brezis, F. Hélein, *Asymptotics for the minimization of a Ginzburg-Landau functional*, Calc. Var., 1 (1993), 123-148.

[10] F. Bethuel, H. Brezis and F. Hélein, *Ginzburg-Landau Vortices*, Birkhäuser, 1997.

[11] F. Bethuel and X. Zheng, *Density of smooth functions between two manifolds in Sobolev spaces*, J. Funct. Anal., 80 (1992), p. 60-75.

[12] A. Boutet de Monvel-Berthier, V. Georgescu and R. Purice, *A boundary value problem related to the Ginzburg-Landau model*, Comm. Math. Phys., 142 (1991), 1-23.

[13] H. Brezis,*Metastable harmonic maps*, in *Metastability and Incompletely Posed Problems,* , S. S. Antman, J. L. Ericksen, D. Kinderlehrer, I. Mller, (eds.) 33-42, Springer-Verlag, 1987.

[14] H. Brezis, *Analyse fonctionnelle*, Masson, 1992.

[15] H. Brezis, *Vorticité de Ginzburg-Landau*, graduate course, Université Paris 6, 2001-2002.

[16] H. Brezis and J.-M. Coron, *Multiple solutions of H-systems and Rellich's conjecture*, Comm. Pure Appl. Math. 37 (1984), 149-187.

[17] H. Brezis and J.-M. Coron, *Large solutions for harmonic maps in two dimensions*, Comm. Math. Phys. 92 (1983), 203-215.

[18] H. Brezis, M. Marcus and I. Shafrir, *Extremal functions for Hardy's inequality with weight*, J. Funct. Anal. 171 (2000), 177-191.

[19] H. Brezis and L. Nirenberg, *Positive Solutions of Nonlinear Elliptic Equations Involving Critical Sobolev Exponents*, Comm. Pure Appl. Math., 36 (1983), 437-477.

[20] H. Brezis and L. Nirenberg, *Degree Theory and BMO*, Part I: *Compact manifolds without boundaries*, Selecta Math., 1 (1995), 197-263 ; Part II: *Compact manifolds with boundaries*, Selecta Math., 2 (1996), 309-368.

[21] M. Comte and P. Mironescu, *Minimizing properties of arbitrary solutions to the Ginzburg-Landau equation*, Proc. Royal Soc. Edinburgh 129A (1999), 1157-1169.

[22] R. J. Donnelly and A. L. Fetter, *Stability of superfluid flow in an annulus*, Phys. Rev. Lett., 17 (1966), 747-750.

[23] O. Druet, *Elliptic equations with critical Sobolev exponent in dimension 3*, Ann. I.H.P., Analyse non-linéaire, 19, (2002), 125-142.

[24] C. M. Elliott, H. Matano and T. Qi, *Zeros of a complex Ginzburg-Landau order parameter with applications to superconductivity*, European J. Appl. Math., 5 (1998), 431 - 448.

[25] H. Federer, *Geometric measure theory*, Springer Verlag, 1969.

[26] D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*, Springer, 1993.

[27] D. Golovaty and L. Berlyand, *On uniqueness of vector-valued minimizers of the Ginzburg-Landau functional in annular domains*, Calc. Var., 14 (2002), 213-232.

[28] S. Jimbo and Y. Morita, *Stability of non constant steady-state solutions to a Ginzburg-Landau equation in higher space dimensions*, Nonlinear Anal. TMA, 22 (1994), 753-770.

[29] S. Jimbo and Y. Morita, *Ginzburg-Landau equations and stable solutions in a rotational domain*, SIAM J. Math. Anal., 27 (1996), 1360-1385.

[30] L. Lassoued and P. Mironescu, *Ginzburg-Landau type energy with discontinuous constraint*, J. d'Analyse, 77 (1999), 1-26.

[31] F. H. Lin and T. Rivière, *Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents*, J. Eur. Math. Soc., 1 (1999), 237-311 ; *Erratum*, 2 (2002), 87-91.

[32] V. Maz'ja, *Sobolev Spaces*, Springer Verlag, 1985.

[33] P. Mironescu, *Explicit bounds for solutions to a Ginzburg- Landau type equation*, Rev. Roumaine Math Pures Appl. 41 (1996), 263-271.

[34] P. Mironescu, *Uniqueness of solutions of the Ginzburg-Landau equations through algebraic methods*, preprint.

[35] T. Rivière, *Line vortices in the $U(1)$-Higgs model*, Control Optim. and Calc. of Var., 1 (1996), 77-167.

[36] H. Triebel, *Theory of function spaces*, Birkhäuser, 1983.

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